Renewable Natural Resources with Tipping Points

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Abstract

Many of the world's renewable resources are in decline. Optimal harvests with smooth recruitment is well studied but in recent years, ecologists have concluded that tipping points in recruitment are common. Recruitment with a tipping point has low-fecundity below the tipping point and high-fecundity above. Under an optimal harvest policy there is always a high-fecundity steady-state. This steady-state is stable but small perturbations may result in large, temporary reductions in the recruitment and harvest rates. In contrast, a low-fecundity steady-state need not exist. When a low-fecundity steady-state exists, there is an endogenous tipping (Skiba) point: below, harvests converge to the low-fecundity steady-state and above, an austere harvest policy transitions the ecosystem to high-fecundity recruitment. If there is hysteresis in recruitment, the high steady-state may not be stable. Moreover, if the high-/low-fecundity differential is large then following a downward perturbation, fecundity optimally remains low.

Key words: renewable resource management, over fishing, tipping point, hysteresis
JEL codes: Q20, Q22, Q23

1 Introduction

Many of the world's renewable resources are in decline, including fisheries (Worm et al., 2006; Jackson, 2008), forests (FAO and UNEP, 2020) and wildlife (Felbab-Brown, 2017). The theoretical literature on modeling renewable resources has a long history, dating back to Gordon (1954), Scott (1955), Smith (1968) and Clark (1973a,b). While the focus of these works has often been on fisheries, the modeling is applicable to renewable resources in general (see Clark, 2010).

The "smooth recruitment function" renewable resource problem is well studied (Clark, 2010), however, it is now believed that many renewable resources are subject to "tipping points" in recruitment. For fisheries, there is a general consensus that tipping points are important (Selkoe et al., 2015; Hunsicker et al., 2018). For instance, minimum population density, genetic diversity, etc. may be required for effective reproduction. On the other hand, the tipping mechanism for tropical rain forests results from changes in rainfall patterns due to deforestation that transitions an ecosystem from rain forest to savanna (Nobre and Borma, 2009; Malhado et al., 2010).

In this paper, I characterize the optimal harvest of a renewable resource in the presence of tipping points. In a model with tipping points, there is always a "good" steady-state but if the high-/low-fecundity differential is small then there may not be a "bad" steady-state. When a bad steady-state exists, there is an endogenous tipping point below which the "standard" harvest policy is optimal and above which, an "austere" harvest policy is optimal. Under an austere harvest policy the resource stock converges to the good steady-state. Without hysteresis in recruitment, the good steady-state is stable and following small perturbations, the resource stock quickly converges back to the good steady-state. But when there is hysteresis in recruitment, the good steady-state may not be stable. First, a small perturbation dropping the resource stock below the tipping point may optimally transition the ecosystem permanently to the bad steady-state. Second, even when recovery of the ecosystem is optimal, there is delay in recovery so that in contrast to the baseline, "no hysteresis" model, the good steady-state is not stable.

In models of renewable resources, the growth rate of the resource stock is the recruitment or reproduction rate less the harvest rate. Recruitment is a function of the resource stock and I model a tipping point as a discontinuity in recruitment where below the tipping point, recruitment has low-fecundity and above, high. Recruitment may also be subject to hysteresis where recruitment is state dependent. In particular, with hysteresis, at low-fecundity, there is a threshold resource stock level required to transition to high-fecundity and at high-fecundity recruitment, there is another, lower threshold below which the ecosystem transitions to low-fecundity (Dudgeon et al., 2010; Selkoe et al., 2015). That is, at intermediate levels of the resource stock, both high and low-fecundity are possible and the current state of fecundity remains unchanged until the tipping point is crossed.

In the absence of hysteresis, there is always a high-fecundity steady-state. This high steady-state is stable in the sense that following a small perturbation, the resource stock will quickly return to it. However, in some cases, the high steady-state coincides with the tipping point and while stable, a small perturbation can result in a large temporary fall in both the recruitment and optimal harvest rates.

Although there is always a high-fecundity steady-state, a low-fecundity steady-state need not exist. First, the stationary point associated with the low-fecundity recruitment function may lie above the tipping point, rendering it infeasible. Second, even when this stationary point is below the tipping point, if the fecundity differential between the high and low-fecundity recruitment functions is relatively small, the optimal harvest is austere and leads to the high-fecundity steady-state.

However, when the low-fecundity stationary point is feasible and the fecundity differential is large, the instantaneous cost of austerity is relatively high. In this case, there is a second, endogenous threshold below the tipping point. If the initial resource stock is above this endogenous tipping point, the optimal harvest policy is austere and leads to the high-fecundity steady-state – even though the instantaneous cost of austerity is high, the length of time this austerity must be borne is relatively low. In contrast, if the initial resource stock is below this endogenous tipping point, the length of time that austerity needs to be maintained is too high and instead, the optimal harvest policy is the standard one, leading to the low-fecundity steady-state.

These results imply that when the initial resource stock is sufficiently high, the optimal harvest will always attain a high-fecundity stationary point even from below the tipping point. But when the difference between high-fecundity and low-fecundity productivity is large and the initial resource stock is small (due perhaps to uncontrolled harvesting), absent an external injection of the renewable resource, the optimal harvest policy does not attain high-fecundity.

With hysteresis on the other hand, the high and low-fecundity recruitment functions overlap and there are two tipping points: a high-fecundity and a low-fecundity tipping point with the former being smaller than the latter. On the high-fecundity recruitment function, if the resource stock falls below the high-fecundity tipping point, recruitment switches to the low-fecundity recruitment function. On the low-fecundity recruitment function, recruitment can only switch to the high-fecundity recruitment function if the resource stock rises to the higher, low-fecundity tipping point.

If the high stationary point coincides with the high-fecundity tipping point then it is no longer stable. A small perturbation can bring the resource stock below the high-fecundity tipping point to low-fecundity recruitment. But now instead of quickly returning to the high-fecundity stationary point, there is either i) significant delay for the resource stock to rise to the (higher) low-fecundity tipping point or ii) returning to high-fecundity is not optimal. The former accords with what we know about the Atlantic northwest cod fishery collapse of the early 1990s. After nearly 30 years of nurturing, the fishery may still not return to sustainable levels until 2025 (Rose and Rowe, 2015).

In the following Section, I describe the model. In Section 3, I examine the optimal renewable resource extraction problem without hysteresis. Next, in Section 4, I discuss how these results change when there is hysteresis in recruitment. Finally, in Section 5, I offer some concluding remarks.

2 The Model

In dynamic models of renewable resources, growth of the resource stock is governed by a recruitment function, $f(x_t)$, where x_t is the resource stock at time t. Given stock x_t, $f(x_t)$ represents the natural growth rate of the resource. In economic terms, the function f is a (biological) production function. In the standard analysis, f is assumed to be a continuous function. In contrast, my interest is in functions that take discrete upward jumps.

In particular, I define the tipping recruitment function as:

$$f(x) = \begin{cases} \pi \tilde{f}(x) & \text{if } x < x_p \\ \tilde{f}(x) & \text{if } x \geq x_p \end{cases} \tag{1}$$

where $f(x)$ is the rate at which the resource stock increases when the stock is $x \geq 0$. The tipping point is $x_p > 0$ and $0 < \pi < 1$, represents the degree to which recruitment productivity is lower when fecundity is low. The function $\tilde{f}$ is strictly increasing, twice differentiable, concave, $\tilde{f}(0) = 0$, $\lim_{x\to 0} \tilde{f}'(x) = \infty$ and $\lim_{x\to\infty} \tilde{f}'(x) = 0$.[1]

At time t, if x_t is the time t resource stock and h_t is the harvest rate then the growth rate of the resource stock is:

$$\dot{x}_t = f(x_t) - h_t. \tag{2}$$

That is, net resource growth is the rate at which the resource is added to the ecosystem, $f(x_t)$, less the harvest rate, h_t (i.e., investment is production less consumption). When harvests are below recruitment ($h_t < f(x_t)$), the resource stock is rising and when harvests exceed recruitment ($h_t > f(x_t)$), the resource stock is falling. At every time t, the harvest rate and the resource stock must be non-negative so that $h_t \geq 0$ and $x_t \geq 0$.

Given recruitment (1), resource stock growth (2), the initial stock of the resource ($x_0 > 0$) and non-negativity constraints, the social welfare function is given by:

$$\int_0^\infty e^{-\rho t} u(h_t) dt \tag{3}$$

where $\rho > 0$ is the social discount rate,[2] $u(h_t)$ is instantaneous social welfare when the harvest rate is h_t. Assume that u is strictly increasing, twice differentiable, concave and $\lim_{h\to 0} u(h) + Ku'(h) = \infty$ for $K > 0$.[3]

[1] With the assumption that $\tilde{f}$ is strictly increasing in the resource stock, the term "maximum sustainable yield" (MSY) is meaningless. This is unrealistic since, in the absence of harvesting, the resource stock would increase without bounds. This can be remedied with the addition of a "predation" and/or "overcrowding" term with the idea that at high resource stock levels, predation and/or crowding become more significant. In particular,

$$f(x) = \begin{cases} \pi\tilde{f}(x) - \delta x & \text{if } x < x_p \\ \tilde{f}(x) - \delta x & \text{if } x \geq x_p \end{cases}$$

where $\delta > 0$ is the predation and/or overcrowding penalty. The δx term is analogous to depreciation in models of economic growth. Since my focus is not on comparisons between MSY and other possible outcomes, for simplicity I assume that $\delta = 0$.

[2] Note that societal patience (willingness to sacrifice current harvests for future harvests) is inversely related to the social discount rate, ρ. Low values of ρ imply high levels of patience and high values of ρ imply low levels of patience.

[3] This condition is stronger than the usual condition that $\lim_{h\to 0} u'(h) = \infty$ but holds for the commonly used family of CRRA utility functions.

An optimal harvest plan solves:

$$\begin{aligned} V(x_0) = \max_{h_t \geq 0} \int_0^\infty e^{-\rho t} u(h_t) dt \\ \text{s.t. } \dot{x}_t = f(x_t) - h_t \\ x_t \geq 0 \\ \text{given } x_0 > 0 \end{aligned} \tag{4}$$

where x_0 is the initial resource stock. The analysis of this problem is complicated by the discontinuity in f.

The current value Hamiltonian for this problem is:

$$\mathscr{H}(x, h, \lambda) = u(h) + \lambda[f(x) - h] \tag{5}$$

where λ is the costate which represents the value of an infinitessimal increase in the resource stock, x. I will proceed to the analysis of (5) in Section 3. In discussing trajectories (optimal and otherwise), it will be useful at times to use the policy function analogue, $h(x)$,[4] that dictates the harvest rate when the resource stock is x.

2.1 Smooth recruitment

Before proceeding, I briefly review the analysis for the simpler case where there is no tipping point and the recruitment function is continuous. In particular, consider the recruitment function, $A\tilde{f}(x)$ where $A > 0$. The solution to (5) satisfies the following necessary conditions:

$$u'(h) = \lambda, \tag{6}$$

$$\dot{\lambda} = \lambda[\rho - A\tilde{f}'(x)], \tag{7}$$

$$\dot{x} = A\tilde{f}(x) - h \tag{8}$$

and the transversality condition:

$$\lim_{t \to \infty} e^{-\rho t} \lambda_t x_t = 0. \tag{9}$$

[4]Strictly speaking, $h(x)$ may be a correspondence rather than a function but in practice, trajectories of interest will have single valued $h(x)$ and for simplicity, I use this imprecise terminology.

Concavity of $\tilde{f}$ and u are sufficient for optimality.

Differentiating (6) with respect to t and using (7) yields:

$$\dot{h} = \frac{1}{\sigma(h)} h[A\tilde{f}'(x) - \rho], \tag{10}$$

where $\sigma(h) = -u''(h)h/u'(h) > 0$. This implies that the optimal harvest rate is increasing when the marginal recruitment rate exceed the social discount rate ($A\tilde{f}'(x) > \rho$) and declining when marginal recruitment rate is less than the social discount rate ($A\tilde{f}'(x) < \rho$).

Together (8) and (10) represent an autonomous system of first-order differential equations that governs the dynamics of the system. Since $\mathscr{H}$ is strictly concave in (x, h), given x_0, there is a unique solution to (8), (10) and (9), $(\hat{x}_t^c, \hat{h}_t^c)$, that converges to steady-state $(\hat{x}^c, \hat{h}^c)$ (Figure 1). Let $\hat{h}^c(x)$ and $V^c(x)$ denote the corresponding policy and value functions. The superscript c denotes variables associated with the continuous problem.

Remark. *Note that both here and subsequently, x and h variables with time subscripts are trajectories, an h with a functional argument is a policy function and a "hat" denotes optimality. A "hat" variable with no time subscript or functional argument is an optimal stationary point.*

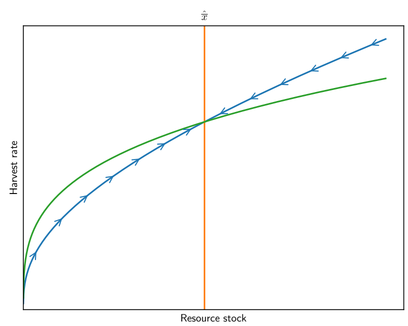
2.2 Austerity

It will be useful to compare trajectories and policies to one another. As a point of comparison, I first define the "standard" policy function, $h^s(x)$. Taking $A \in \{\pi, 1\}$, let $\underline{f}(\cdot) = \pi\tilde{f}(\cdot)$ and $\overline{f}(\cdot) = \tilde{f}(\cdot)$ be continuous low- and high-fecundity, recruitment functions and denote the corresponding current value Hamiltonians as $\underline{\mathscr{H}}$ and $\overline{\mathscr{H}}$. Given x_0, let the unique, optimal trajectories be given by $(\hat{\underline{x}}_t, \hat{\underline{h}}_t)$ and $(\hat{\overline{x}}_t, \hat{\overline{h}}_t)$ for $t \geq 0$. These trajectories converge to steady-states $(\hat{\underline{x}}, \hat{\underline{h}})$ and $(\hat{\overline{x}}, \hat{\overline{h}})$ and have corresponding policy functions $\hat{\underline{h}}(x)$ and $\hat{\overline{h}}(x)$ and value functions $\underline{V}(x)$ and $\overline{V}(x)$. For $\tilde{f}$ and u that satisfy the assumed properties, given resource stock $x > 0$, it must be the case that $\hat{\underline{h}}(x) < \hat{\overline{h}}(x)$ and $\underline{V}(x) < \overline{V}(x)$.

Define:

$$h^s(x) = \begin{cases} \underline{h}(x) & \text{if } x < x_p \\ \overline{h}(x) & \text{if } x \geq x_p \end{cases} \tag{11}$$

Figure 1: Smooth recruitment optimal harvest



Legend for Figures 1 to 4

$h^s(x)$
$h(x)$
$\dot{x} = 0$
$\dot{h} = 0$

Call $(\underline{\hat{x}}, \underline{\hat{h}})$ and $(\hat{\bar{x}}, \hat{\bar{h}})$ the *notional* low and high steady-states and $h^s(x)$ the *standard policy.*

Having defined the standard policy, I now define "austerity." Loosely speaking, I consider a trajectory (x_t, h_t) to be austere if it lies below the standard policy, $h^s(x)$. To be more precise:

Definition 1. *Harvest policy* $h(x)$ *is* austere *relative to* $h^0(x)$ *if* $h(x) \leq h^0(x)$ *and there is* $x' < x''$ *such that when* $x \in [x', x'')$, $h(x) < h^0(x)$.

Definition 2. *Trajectory* (x_t, h_t) *is* austere *relative to harvest policy* $h^0(x)$ *if the corresponding harvest policy,* $h(x)$*, defined over the domain* $[\inf\{x_t\}_{t=0}^{\infty}, \sup\{x_t\}_{t=0}^{\infty}]$ *is austere relative to* $h^0(x)$ *over the same domain.*

Definition 3. *Harvest policy* $h(x)$ *or trajectory* (x_t, h_t) *is* austere *if it is austere relative to* $h^s(x)$.

We will see that under the tipping recruitment function, $f(x)$, the optimal trajectory, $(\hat{x}_t, \hat{h}_t)$, may be austere and the notional stationary points for the low- and high-fecundity continuous problems, $(\underline{\hat{x}}, \underline{\hat{h}})$ and $(\hat{\bar{x}}, \hat{\bar{h}})$, need not be stationary for the discontinuous, tipping problem.

3 Optimal harvest

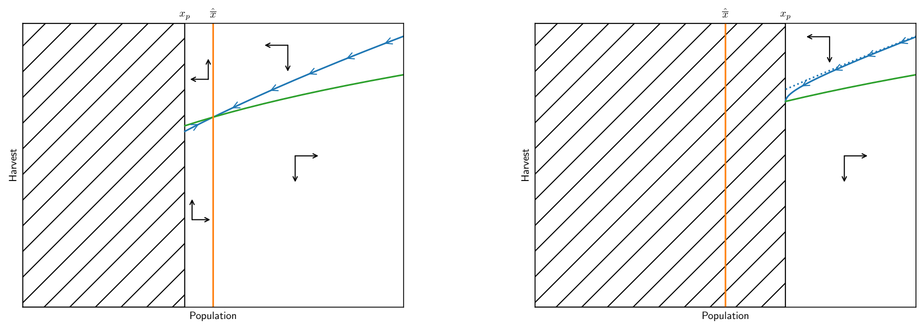
I solve the optimal discontinuous renewable resource problem by construction. I begin with the solution to the problem where $x_0 \geq x_p$ and assume that the resource stock is constrained to remain at or above the tipping point (i.e., $x_t \geq x_p$ for any $t \geq 0$). The solution to this problem will yield constrained optimal path $(\hat{x}_t^*, \hat{h}_t^*)$ that converges to $(\hat{x}^*, \hat{h}^*)$ and value function $V^*(x)$ for $x \geq x_p$.[5]

Next, I solve the problem for $x_0 < x_p$, allowing for the possibility that the optimal trajectory may transition to the high-fecundity recruitment function. To solve this, note that there are two possibilities: i) the optimal trajectory, $(\hat{x}_{*t}, \hat{h}_{*t})$, converges to the low notional stationary point, $(\hat{x}_*, \hat{h}_*) = (\underline{\hat{x}}, \underline{\hat{h}})$ or ii) the optimal trajectory reaches the tipping point so that $\hat{x}_{*T} = x_p$ at time T

[5]The superscript *, here and subsequently, is used to denote variables and functions associated with the constrained, high-fecundity problem. Similarly, a subscript * will be used to denote variables and functions associated with the corresponding low-fecundity problem.

Figure 2: Constrained upper problem

(a) Constrained interior (b) Constrained boundary



with terminal value $e^{-\rho T}V^*(x_p)$ under the implicit assumption that harvests thereafter follow the constrained, high-fecundity solution that converges to $(\hat{x}^*, \hat{h}^*)$.

Finally, given these solutions, I show that the constrained, high-fecundity solution is optimal when $x_0 \geq x_p$ and x_t is unconstrained. Consequently, the solution to the low-fecundity problem is also optimal.

3.1 Constrained high-fecundity problem

Consider the problem where the resource stock is constrained to stay at or above the tipping point:

$$\begin{aligned} V^*(x_0) = \max_{h_t \geq 0} \int_0^\infty e^{-\rho t} u(h_t) dt \\ \text{s.t. } \dot{x}_t = \tilde{f}(x_t) - h_t \\ x_t \geq x_p \\ \text{given } x_0 \geq x_p. \end{aligned} \tag{12}$$

This is problem (4) for $x_0 \geq x_p$ and where there is the additional constraint that $x_t \geq x_p$ for all $t \geq 0$.

Proposition 1. *For the high-fecundity problem given by (12), the optimal trajectory, $(\hat{x}_t^*, \hat{h}_t^*)$ for all $t \geq 0$, is unique and:*

i) if $\hat{\bar{x}} \geq x_p$ then $(\hat{x}_t^*, \hat{h}_t^*) = (x_t^s, h_t^s)$ and $\lim_{t\to\infty} \hat{x}_t^* = \hat{\bar{x}}$,

ii) if $\hat{\bar{x}} < x_p$ then $(\hat{x}_t^*, \hat{h}_t^*)$ is austere and there is some time $\tau < \infty$ such that for all $t \geq \tau$, $\hat{x}_t^* = x_p$ and

iii) for all $x \geq x_p$, $V^*(x) \geq \frac{u(\tilde{f}(x_p))}{\rho}$.

The proof of Proposition 1 and all subsequent proofs are in given in Appendix A.

When the high notional stationary point is at least the tipping point ($\hat{\bar{x}} \geq x_p$), the constraint is non-binding and the optimal policy corresponds to the standard policy and the constrained optimal stationary point corresponds to the high notional stationary point defined in Section 2.2 (see Figure 2a).

But if the high notional stationary point is below the tipping point ($\hat{\bar{x}} < x_p$), the constraint is binding and the optimal harvest is austere with the resource stock falling and stopping at $x_\tau = x_p$ at time τ (see Figure 2b). When the harvest trajectory is not sufficiently austere, the resource stock reaches the tipping point too quickly. When the harvest trajectory is too austere, the trajectory crosses the $\dot{x} = 0$ line and is suboptimal since harvests decline over time but harvesting $\tilde{f}(x_p)$ is always feasible.

3.2 Low-fecundity problem

Now consider the low fecundity problem where $x_0 < x_p$. Assume that should trajectory (x_t, h_t) reach the tipping point, x_p, at some time T, there is a terminal payoff $e^{-\rho T} V^*(x_p)$. That is, after time T, the trajectory is implicitly assumed to follow the solution from Section 3.1.

For this problem there are two candidate outcomes. In one outcome, high-fecundity is not attained and the resource stock converges to the low notional steady-state, $\underline{\hat{x}}$. In the second outcome, the resource stock increases ($\pi\tilde{f}(x) - h > 0$) until it reaches the tipping point, x_p, and recruitment becomes high-fecundity at time T.

The optimization problem for the latter type of outcome is:

$$\begin{aligned} V_2(x_0) = \max_{h_t \geq 0} \int_0^T e^{-\rho t} u(h_t) dt + e^{-\rho T} V^*(x_p) \\ \text{s.t. } \dot{x}_t = \pi \tilde{f}(x_t) - h_t \\ x_t \geq 0 \\ x_T = x_p \\ T \text{ free} \\ \text{given } x_0 < x_p. \end{aligned} \tag{13}$$

This is a control problem with fixed terminal point, x_p, "scrap value," $e^{-\rho T} V^*(x_p)$, and free terminal time, T.

Either type of outcome must solve the current value Hamiltonian (5) so that both types of outcomes must satisfy (8) and (10). Since $x < x_p$, $f(x) = \pi \tilde{f}(x)$ and $f'(x) = \pi \tilde{f}'(x)$. In addition, optimal trajectories must satisfy the appropriate transversality conditions. For the outcome that converges to the low notional stationary point, this is the standard transversality condition (9). For the outcome that transitions to the high-fecundity recruitment function at time T, the transversality condition is:

$$\lim_{t \to T} \underline{\mathscr{H}}(x_t, h_t, \lambda_t) = \rho V^*(x_p). \tag{14}$$

This has the intuitive interpretation that as $t \to T$, the flow value of trajectory (x_t, h_t), as represented by the current value Hamiltonian, must be equal to the flow value of the terminal payoff, $\rho V^*(x_p)$. Since $h_t < \pi \tilde{f}(x_t)$, Lemma 1 from the Appendix shows that $\underline{\mathscr{H}}(x, h, u'(h))$ is decreasing in h and implies that if $\lim_{t \to T} \underline{\mathscr{H}}(x_t, h_t, \lambda_t) > \rho V^*(x_p)$ then high-fecundity is being attained too quickly so that (x_t, h_t) is overly austere and a higher harvest rate would be welfare improving. Conversely, if $\lim_{t \to T} \underline{\mathscr{H}}(x_t, h_t, \lambda_t) < \rho V^*(x_p)$ then the transition to high-fecundity is too slow and (x_t, h_t) is insufficiently austere so that a lower harvest rate is optimal.

The overall solution to the low-fecundity problem will have an endogenous tipping point, below which the standard trajectory obtains and above which a trajectory reaching x_p and high fecundity is attained.

Proposition 2. *For the low-fecundity problem, if* $\underline{\hat{h}}(x_p) \leq \tilde{f}(x_p)$ *then the optimal trajectory,* $(\hat{x}_{*t}, \hat{h}_{*t})$ *for all* $t \geq 0$*, is unique and there exists* $x'_p \in [0, x_p)$ *such that* *i) if $x_0 < x_p'$ then $(\hat{x}_{*t}, \hat{h}_{*t}) = (x_t^s, h_t^s)$ and $\lim_{t\to\infty} \hat{x}_{*t} = \underline{\hat{x}}$,*

ii) if $x_0 > x_p'$ then $(\hat{x}_{*t}, \hat{h}_{*t})$ is austere and there is some time $\tau < \infty$ such that $\hat{x}_{*\tau} = x_p$.

There is an endogenous tipping point, x_p' (possibly 0). Above x_p', the optimal trajectory is austere and at time τ, $x_\tau = x_p$ and thereafter, the optimal trajectory follows the high-fecundity solution from Section 3.1. Below x_p', austerity is too costly and the optimal harvest is the standard policy which converges to the low notional steady-state, $\underline{\hat{x}}$.

The condition that $\underline{\hat{h}}(x_p) \leq \tilde{f}(x_p)$ implies that the standard harvest policy for $x < x_p$ is bounded above by high-fecundity recruitment and allows for the easy ranking of the high- and low-fecundity value functions at $x = x_p$. This can be violated when π is close to 1 and recruitment is close to being continuous.

3.3 Unconstrained optimality

Now consider the full problem where the resource stock is only constrained to be non-negative. In particular, when $x_0 \geq x_p$, a permissible trajectory can have $x_t < x_p$ for some $t > 0$. Let $(\hat{x}_t, \hat{h}_t)$ be the trajectory that solves this problem.

Proposition 3. *If $\underline{\hat{h}}(x_p) \leq \tilde{f}(x_p)$ then the optimal trajectory for the unconstrained problem, $(\hat{x}_t, \hat{h}_t)$ for all $t \geq 0$, is unique and there exists $x_p' \in [0, x_p)$ such that*

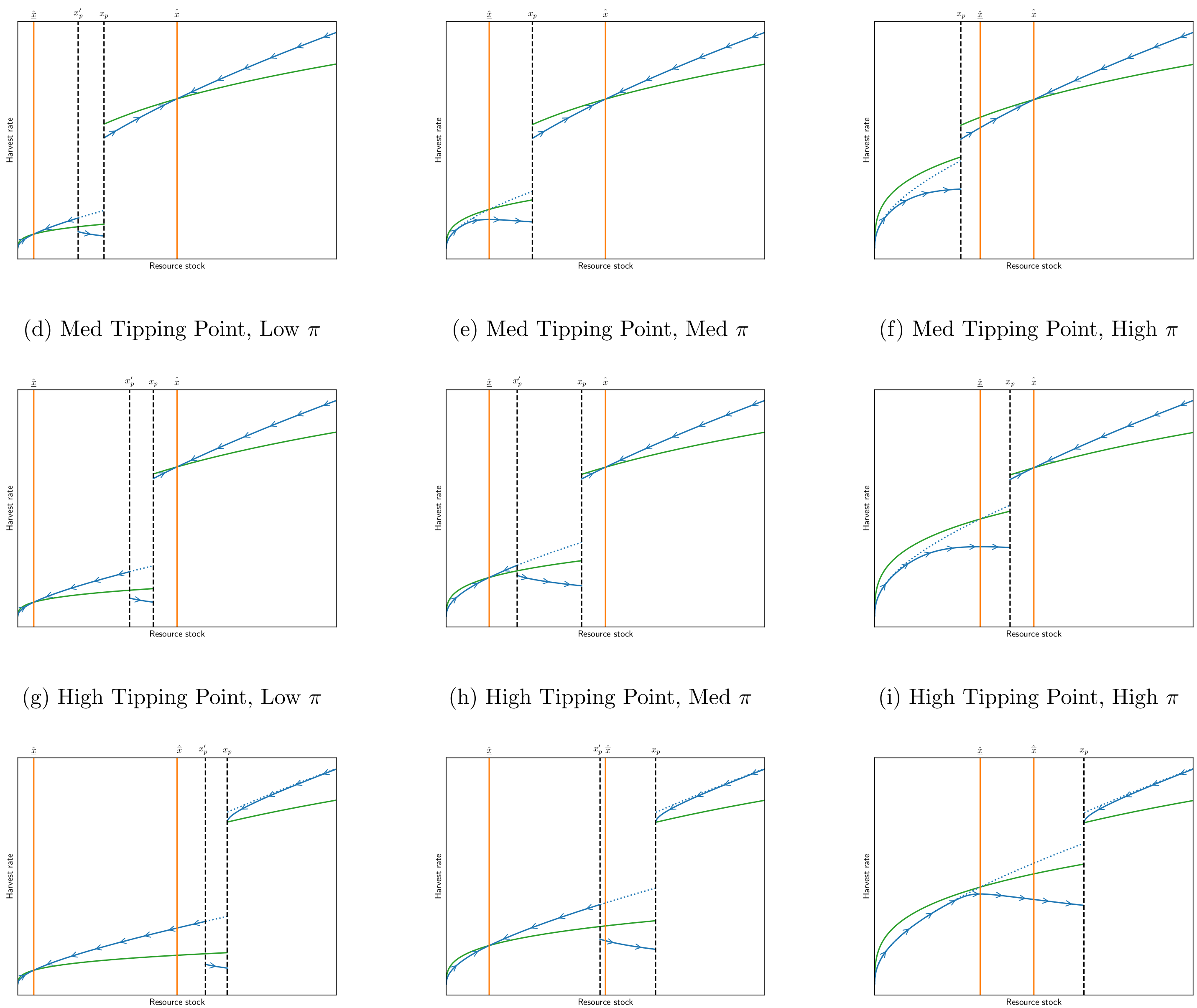
i) if $x_0 \in (0, x_p')$ then the optimal trajectory is $(\hat{x}_t, \hat{h}_t) = (x_t^s, h_t^s)$ and $\lim_{t\to\infty} \hat{x}_t = \underline{\hat{x}}$,

ii) if $x_0 \in [x_p', x_p)$ then the optimal trajectory $(\hat{x}_t, \hat{h}_t)$ is austere and there exists $\tau < \infty$ such that $\hat{x}_\tau = x_p$;
when $\bar{\hat{x}} > x_p$, $\lim_{t\to\infty} \hat{x}_t = \bar{\hat{x}}$;
when $\bar{\hat{x}} \leq x_p$, $\hat{x}_t = x_p$ for $t \geq \tau$,

iii) if $x_0 \geq x_p$ and $\bar{\hat{x}} \geq x_p$ then $(\hat{x}_t, \hat{h}_t) = (\hat{x}_t^s, \hat{h}_t^s)$ and $\lim_{t\to\infty} \hat{x}_t = \bar{\hat{x}}$,

iv) if $x_0 \geq x_p$ and $\bar{\hat{x}} < x_p$ then $(\hat{x}_t, \hat{h}_t)$ is austere and there exists $\tau < \infty$ such that $\hat{x}_t = x_p$ for $t \geq \tau$.

Figure 3: Optimal harvest policies



(a) Low Tipping Point, Low π

(b) Low Tipping Point, Med π

(c) Low Tipping Point, High π

(d) Med Tipping Point, Low π

(e) Med Tipping Point, Med π

(f) Med Tipping Point, High π

(g) High Tipping Point, Low π

(h) High Tipping Point, Med π

(i) High Tipping Point, High π

Above the tipping point ($x_0 \geq x_p$) there is always a stable, high-fecundity steady-state. When $\hat{\bar{x}} \geq x_p$, the tipping point is non-binding and it is the continuous, high-fecundity stationary point ($\hat{x} = \hat{\bar{x}}$) so that for $x_0 \geq x_p$, $(\hat{x}_t, \hat{h}_t) = (\hat{x}_t^s, \hat{h}_t^s)$ (see Figures 3a to 3f). But if $\hat{\bar{x}} < x_p$, then the high-fecundity steady-state occurs at the tipping point so that $\hat{x} = x_p$. In order to reach this steady-state optimally, harvests must be austere – the standard harvest policy will either bring the resource stock below the tipping point or else bring it to the tipping point too quickly. Even though a stationary point where $\hat{x} = x_p$ is stable, a small perturbation in the resource stock can result in a large fall in both the harvest rate and the resource recruitment rate (see Figures 3g to 3i).

Below the tipping point ($x_0 < x_p$), there is a range of resource stocks under which a policy leading to recovery of the ecosystem is optimal. However, there is an endogenous tipping point, x_p', below which the cost of austerity is too high and recovery, while feasible, becomes suboptimal and instead the resource stock converges to the low-fecundity stationary point, $\underline{\hat{x}}$ (Figures 3a, 3d, 3e, 3g and 3h). That is, given the discount rate, ρ, and other model parameters, below this endogenous tipping point, an austere harvest policy is inferior to the standard policy and has the low steady-state, $(\underline{\hat{x}}, \underline{\hat{h}})$. In contrast, above this endogenous tipping point, although the instantaneous cost of austerity is high, it does not need to be borne for long and the optimal harvest transitions the ecosystem to high-fecundity.

Finally, it may be that the endogenous tipping point is inconsequential ($x_p' = 0$). For instance, when the low notional steady-state is infeasible ($\underline{\hat{x}} > x_p$) (Figures 3b and 3c) or when the high-/low-fecundity differential is relatively small (Figures 3f and 3i) then there is no non-trivial low steady-state. That is, as long as $x_0 > 0$, the optimal trajectory always reaches the high-fecundity steady-state, $\hat{x}^* = \max\{\hat{\bar{x}}, x_p\}$. Despite the potential nonexistence of a bad long term outcome, as practitioners, we will be more interested in parameter configurations where the tipping point has a significant and tangible impact.

4 Hysteresis

In recent years there is evidence that recovery from environmental damage can be subject to hysteresis (Field et al. 2007; Storlazzi et al. 2009 for coral reefs, Lindig-Cisneros et al. 2003 for wetlands, Hirota et al. 2011 for rain forests). For instance, hysteresis has become an important factor that marine ecologists consider as they seek to understand tipping points (Selkoe et al., 2015).

When recruitment is subject to hysteresis, there are two tipping points. If fecundity is high then the tipping point is given by x_p. If the resource stock falls below this tipping point, the ecosystem switches to low-fecundity. With hysteresis, a higher tipping point must be reached in order for the ecosystem to transition to high-fecundity – the low-fecundity tipping point is given by $x_p^{\hbar} > x_p$. Functionally, the hysteretic recruitment function has a second state, s:

$$f(x, s) = (1 - s)\pi \tilde{f}(x) + s\tilde{f}(x)$$

where $s \in \{0, 1\}$ with $s = 1$ representing the high-fecundity state and $s = 0$ the low-fecundity state. For $x < x_p$, $s = 0$, for $x \geq x_p^{\hbar}$, $s = 1$ and for $x \in [x_p, x_p^{\hbar})$, $\dot{s} = 0$ (i.e., s retains its value). Recruitment can change discontinuously at x_p and $x_p^{\hbar}$.

In the model with hysteresis, denote the optimal trajectory $(\hat{x}_t^{\hbar}, \hat{h}_t^{\hbar})$:

Proposition 4. *Assume there is hysteresis in recruitment. If $\underline{\hat{h}}(x_p^{\hbar}) \leq \tilde{f}(x_p^{\hbar})$ then the optimal trajectory, $(\hat{x}_t^{\hbar}, \hat{h}_t^{\hbar})$ for $t \geq 0$, is unique and there exists $x_p^{\hbar\prime} \in [0, x_p^{\hbar})$ such that*

i) if $x_0 \in (0, x_p^{\hbar\prime})$ then the optimal trajectory is $(\hat{x}_t^{\hbar}, \hat{h}_t^{\hbar}) = (\hat{x}_t^s, \hat{h}_t^s)$ and $\lim_{t\to\infty} \hat{x}_t^{\hbar} = \underline{\hat{x}}$,

ii) if $x_0 \in [x_p^{\hbar\prime}, x_p^{\hbar})$ and $s_0 = 0$ then the optimal trajectory $(\hat{x}_t^{\hbar}, \hat{h}_t^{\hbar})$ is austere and there exists $\tau < \infty$ such that $\hat{x}_\tau^{\hbar} = x_p^{\hbar}$ and

when $\hat{\bar{x}} > x_p$, $(\hat{x}_t^{\hbar}, \hat{h}_t^{\hbar}) = (x_t^s, h_t^s)$ for $t \geq \tau$ and $\lim_{t\to\infty} \hat{x}_t^{\hbar} = \hat{\bar{x}}$,

when $\hat{\bar{x}} \leq x_p$, $(\hat{x}_t^{\hbar}, \hat{h}_t^{\hbar})$ is such that there exists $\tau' > \tau$ such that $\hat{x}_t = x_p$ for $t \geq \tau'$,

iii) if $x_0, \hat{\bar{x}} \geq x_p$ and $s_0 = 1$ then the optimal trajectory is $(\hat{x}_t^{\hbar}, \hat{h}_t^{\hbar}) = (\hat{x}_t^s, \hat{h}_t^s)$ and $\lim_{t\to\infty} \hat{x}_t^{\hbar} = \hat{\bar{x}}$,

iv) if $x_0 \geq x_p > \hat{\bar{x}}$ and $s_0 = 1$ then the optimal trajectory $(\hat{x}_t^{\hbar}, \hat{h}_t^{\hbar})$ is austere and there exists $\tau < \infty$ such that $\hat{x}_t^{\hbar} = x_p$ for $t \geq \tau$.

v) $x_p' < x_p^{\hbar\prime}$.

As in the model without hysteresis, for low-fecundity recruitment there is an endogenous tipping point, below which an austere harvest achieving high-fecundity recruitment is suboptimal and instead the optimal harvest follows the standard policy converging to the low-fecundity steady-state (see Figures 4a, 4b, 4d, 4e and 4g to 4i). Above the endogenous tipping point, the cost of austerity is relatively low and as such the optimal harvest policy is austere and attains high-fecundity recruitment (Figure 4). Since the low-fecundity tipping point is greater than the high-fecundity tipping point, upon reaching high-fecundity recruitment, the optimal harvest policy may then reverse course and spend down the resource stock to reach the high-fecundity steady-state (Figures 4d to 4i).

When the high-fecundity steady-state corresponds to the high-fecundity tipping point, the steady-state is no longer stable. If the high and low-fecundity differential is not too large then a perturbation that drops the resource stock below the tipping point requires an extended recovery period to return to high-fecundity, whereupon the optimal harvest spends down the resource stock to return to the steady-state (see Figures 4h and 4i). However, if the high- and low-fecundity differential is large then the endogenous tipping point, $x_p^{\hbar\prime}$, may be greater than the exogenous, high-fecundity tipping point. While returning to high-fecundity recruitment is feasible, it is suboptimal so that falling below the tipping point is permanent (Figure 4g). That is, a high-fecundity stationary point that corresponds to the high-fecundity tipping point is not even "long run stable."

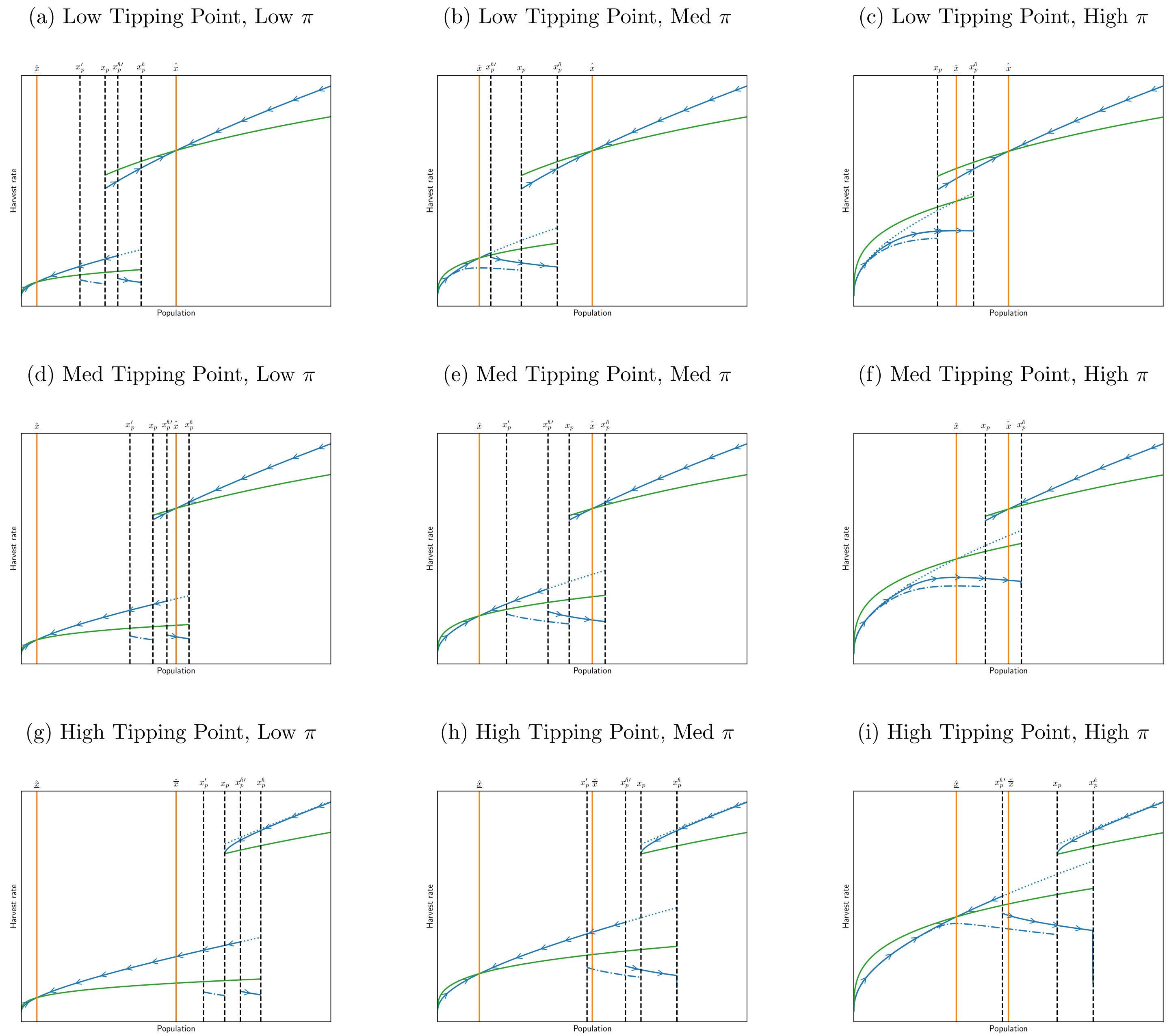
In a fishery context, a clear illustration of a slow recovery from an ecosystem collapse is the case of the Atlantic northern cod fishery in the early 1990s (Hutchings and Myers, 1994; Walters and Maguire, 1996). After nearly 30 years, the fishery has still not recovered to sustainable levels (Sguotti et al., 2019) and may not do so until 2025 (Rose and Rowe, 2015).

5 Conclusion

In this paper I characterized the optimal extraction of a renewable resource where recruitment is subject to tipping points.

In the baseline model with no hysteresis and a single tipping point, there is always a high-fecundity steady-state. The steady-state resource stock is stable but if it coincides with the tipping point, recruitment and harvest rates are not stable.

Figure 4: Hysteretic optimal harvest policies



(a) Low Tipping Point, Low π

(b) Low Tipping Point, Med π

(c) Low Tipping Point, High π

(d) Med Tipping Point, Low π

(e) Med Tipping Point, Med π

(f) Med Tipping Point, High π

(g) High Tipping Point, Low π

(h) High Tipping Point, Med π

(i) High Tipping Point, High π

In contrast, a low-fecundity steady-state does not always exist. There is an endogenous tipping (Skiba) point: below this tipping point, the optimal harvest is the standard policy that converges to the notional, low-fecundity steady-state; above this tipping point, the optimal harvest is austere and attains high-fecundity and eventually the high-fecundity steady-state. If this endogenous tipping point is trivial then there is no low-fecundity steady-state and the high-fecundity steady-state is unique.

If there is hysteresis in recruitment then there are two tipping points and the high and low-fecundity recruitment functions overlap so that recruitment is history dependent. Resource stocks above the high-fecundity tipping point will continue high-fecundity recruitment unless the resource stock falls below the high-fecundity tipping point. Resource stocks below the low-fecundity tipping point will continue with low-fecundity unless the resource stock rises to the low-fecundity tipping point. As with the baseline tipping point model, there is always a high-fecundity steady-state and there is an endogenous tipping point. One key difference is that a high-fecundity steady-state that coincides with the high-fecundity tipping point is not stable. When the high and low-fecundity differential is small, a perturbation that brings the resource stock below the high-fecundity tipping point results in a time-consuming climb to the low-fecundity tipping point and the eventual return to high-fecundity. But when the high and low-fecundity differential is large, a fall below the high-fecundity tipping point is permanent, not because returning to high-fecundity is impossible but because it is suboptimal.

With overharvesting (e.g., a large downward perturbation), the resource stocks can easily fall below the high-fecundity tipping point. Without hysteresis, if the fall is not too severe, the ecosystem will eventually be nursed back to health. But with hysteresis, if the fecundity differential is large then recovery is suboptimal and low-fecundity becomes permanent.

An important point of this analysis is that under the assumptions of the model, it is always feasible to attain high-fecundity recruitment. However, attaining high-fecundity is not always optimal. Optimality will depend on the location of tipping point(s), fecundity differentials and the level of societal patience.

Finally, it is worth emphasizing that this analysis has focused only on optimal harvest policy. In practice, many factors must be accounted for in order to implement an optimal policy. This can include the commons problem where even when the optimal harvest is known, individual incentives lead to overharvest. The commons problem can be further complicated by the structure of the resource market, leading to possibly second-best harvest policy implementations. Moreover, there may be important interactions between different renewable resources. For instance, deforestation results in the loss of habitat for native wildlife that may impact wildlife fecundity. These complications, while important, are not considered here and call for further research.

Appendix

A Proofs

Lemma 1.

$$\mathcal{H}(x, h, u'(h)) = u(h) + u'(h)[A\tilde{f}(x) - h] \tag{A.1}$$

is strictly decreasing in h when $h < A\tilde{f}(x)$ and strictly increasing in h when $h > A\tilde{f}(x)$.

Proof. Differentiating with respect to h,

$$\frac{\partial \mathcal{H}(x, h, u'(h))}{\partial h} = u''(h)[A\tilde{f}(x) - h].$$

Since $u''(h) < 0$, this is negative when $h < A\tilde{f}(x)$ and positive when $h > A\tilde{f}(x)$. □

Proof of Proposition 1. i) If $\hat{\bar{x}} \geq x_p$ then $(\hat{\bar{x}}_t, \hat{\bar{h}}_t)$ is uniquely optimal since the constraint $x_t \geq x_p$ is non-binding and thus $(\hat{x}^*_t, \hat{h}^*_t) = (\hat{\bar{x}}_t, \hat{\bar{h}}_t) = (x^s_t, h^s_t)$ and $\hat{x}^*_t \to \hat{\bar{x}}$.

ii) For $\hat{\bar{x}} < x_p$, there are two classes of trajectories (x_t, h_t) satisfying (8) and (10).

In one, (x_t, h_t) crosses the $\dot{x} = 0$ curve (green in fig. 2b). Since $\hat{\bar{x}} < x_p$, the crossing point cannot be stationary ($\dot{h}_t < 0$) and standard arguments show that (x_t, h_t) is suboptimal.

In the second class, (x_t, h_t) reaches $x_T = x_p$ at some time T. For such trajectories, upon reaching $x_T = x_p$, (8) and (10) imply that $(x_t, h_t) = (x_p, \tilde{f}(x_p))$ for $t > T$, otherwise the constraint that $x_t \geq x_p$ would be violated. The optimal harvest problem can thus be rewritten as a free-terminal-time problem with terminal value, $e^{-\rho T}u(\tilde{f}(x_p))/\rho$:

$$V^*(x_0) = \max_{h_t \geq 0} \int_0^T e^{-\rho t} u(x_t) dt + e^{-\rho T} \frac{u(\tilde{f}(x_p))}{\rho}$$
$$\text{s.t. } \dot{x}_t = \tilde{f}(x_t) - h_t$$
$$x_t \geq x_p$$
$$T \text{ free}$$
$$\text{given } x_0 \geq x_p$$

where the transversality condition is

$$\overline{\mathscr{H}}(x_T, h_T, \lambda_T) = u(\tilde{f}(x_p)).^{6} \tag{A.2}$$

But $u(h_T) + \lambda[\tilde{f}(x_p) - h_T] = u(\tilde{f}(x_p))$ if and only if $h_T = \tilde{f}(x_p)$. Therefore, the optimal trajectory has $(x_T, h_T) = (x_p, \tilde{f}(x_p))$ so that $(\hat{x}_t^*, \hat{h}_t^*) = (x_t, h_t)$ for $t < T$ where $(\hat{x}_t^*, \hat{h}_t^*) = (x_p, \tilde{f}(x_p))$ for $t \geq T$. Let $\tau = T$. Since $(\hat{x}_t^*, \hat{h}_t^*) = (x_t, h_t)$ is the only trajectory satisfying (A.2), the solution is unique.

Now consider trajectory (x_t^s, h_t^s) that corresponds to the standard policy $h^s(\cdot)$ for a given x_0. In the continuous, high-fecundity problem, $x_t^s \to \hat{\overline{x}}$ from above so that at some time T', $x_{T'}^s = x_p$ and $h_{T'}^s > \tilde{f}(x_p)$. Therefore, $(\hat{x}_t^*, \hat{h}_t^*)$ and is austere.

iii) Since $\tilde{f}$ is strictly increasing, if $x_t \geq x_p$ for any $t \geq 0$ then harvesting $\tilde{f}(x_p)$ is always feasible but not necessarily optimal. Therefore

$$\begin{aligned} V^*(x_0) &= \int_0^\infty e^{-\rho t} u(\hat{h}_t^*) dt \\ &\geq \int_0^\infty e^{-\rho t} u(\tilde{f}(x_p)) dt \\ &= \frac{u(\tilde{f}(x_p))}{\rho}. \end{aligned} \tag{A.3}$$

□

[6] See p. 11 for a discussion of the intuition behind this transversality condition.

Proof of Proposition 2. The plan of the proof is to: 1. Characterize candidate optimal trajectories: a) (x_{1t}, h_{1t}) such that $\lim_{t\to\infty}(x_{1t}, h_{1t}) = (\underline{\hat{x}}, \underline{\hat{h}})$ and b) (x_{2t}, h_{2t}) where there exists some $\tau > 0$ such that $x_{2\tau} = x_p$, 2. Show that (x_{2t}, h_{2t}) is austere and 3. Show that there is an endogenous tipping point, x'_p, below which (x_{1t}, h_{1t}) is optimal and above which (x_{2t}, h_{2t}) is optimal.

1. a) Begin by considering trajectories, (x_{1t}, h_{1t}), that converge to the low notional steady-state, $(\underline{\hat{x}}, \underline{\hat{h}})$.

   If $\underline{\hat{x}} < x_p$ then for $A = \pi$, the standard analysis shows that for $x_0 < x_p$, $(\underline{\hat{x}}_t, \underline{\hat{h}}_t)$ is the unique trajectory satisfying (8), (10) and (9) and converges to $(\underline{\hat{x}}, \underline{\hat{h}})$. Let $(x_{1t}, h_{1t}) = (\underline{\hat{x}}_t, \underline{\hat{h}}_t)$ and denote the corresponding value function $V_1(x) = \underline{V}(x)$.

   If $\underline{\hat{x}} \geq x_p$ then $\underline{\hat{x}}$ is not feasible and cannot be a steady-state. Therefore there is no trajectory satisfying (8), (10) and(9) that converges to $(\underline{\hat{x}}, \underline{\hat{h}})$) (see fig. 3c).

   b) Now consider a trajectory (x_t, h_t) such that at some time T, $x_T = x_p$. In order to reach x_p from $x_0 < x_p$, it must be that $\dot{x}_t > 0$. For $A = \pi$, (8) implies that $h_t < \pi\tilde{f}(x_t)$ and consequently, $\lim_{t\to T} h_t \leq \pi\tilde{f}(x_p)$; let $h_T^- = \lim_{t\to T} h_t$. Upon reaching $x_T = x_p$, the terminal payoff is $e^{-\rho T}V^*(x_p)$.

   Given the terminal payoff $e^{-\rho T}V^*(x_p)$, the optimal trajectory minimizes the time required to reach x_p while not sacrificing too much through austere early harvests. This balance is captured by the free-stopping-time transversality condition, (14).

   For $h = \pi\tilde{f}(x_p)$,

$$\begin{aligned}\underline{\mathcal{H}}(x_p, h, \lambda) &= u(\pi\tilde{f}(x_p)) \\ &< u(\tilde{f}(x_p)) = \rho\frac{u(\tilde{f}(x_p))}{\rho} \\ &\leq \rho V^*(x_p). & \text{Proposition 1}\end{aligned}$$

   Since $\lim_{h\downarrow 0} u(h) + \pi\tilde{f}(x_p)u'(h) = \infty$ (by assumption), $\lim_{h\downarrow 0} \underline{\mathcal{H}}(x_p, h, u'(h)) = \infty$. Continuity implies that there is at least one h_T^- such that (14) is satisfied.

   From Lemma 1, since $h < \pi\tilde{f}(x)$, $\underline{\mathcal{H}}(x, h, u'(h))$ is strictly decreasing in h and therefore there is a unique $h_T^- \in (0, \pi\tilde{f}(x_p))$ such that (8),

(10) and (14) hold. Let (x_{2t}, h_{2t}) be the trajectory satisfying (8), (10), (14) such that $x_{2T} = x_p$. Let $\tau = T$.

2. To show that (x_{2t}, h_{2t}) is austere, consider 2 cases: a) $\underline{\hat{x}} \geq x_p$ and b) $\underline{\hat{x}} < x_p$.

   a) For $\underline{\hat{x}} \geq x_p$, we know that $\hat{\overline{x}} > x_p$ so that the standard analysis holds for the high-fecundity problem and $\hat{h}^*(x_p) = \hat{\overline{h}}(x_p)$ and $V^*(x_p) = \overline{V}(x_p)$. Since $\overline{V}(x) > \underline{V}(x)$ it follows that:

   $$\begin{aligned}\underline{\mathcal{H}}(x_p, h_{2\tau}^-, u'(h_{2\tau}^-)) &= \rho V^*(x_p) && \text{transversality}\\ &= \rho\overline{V}(x_p)\\ &> \rho\underline{V}(x_p)\\ &= \underline{\mathcal{H}}(x_p, \underline{\hat{h}}(x_p), u'(\underline{\hat{h}}(x_p))) && \text{HJB.}\end{aligned}$$

   Since $h_{2\tau}^-, \underline{\hat{h}}(x_p) \leq \pi\tilde{f}(x_p)$ and $\underline{\mathcal{H}}(x_p, h, u'(h))$ is decreasing in h when $h \leq \pi\tilde{f}(x)$ (Lemma 1), it must be that $h_{2\tau}^- < \underline{\hat{h}}(x_p)$. Therefore, (x_{2t}, h_{2t}) is austere.

   b) For $\underline{\hat{x}} < x_p$, if $x_0 \in (\underline{\hat{x}}, x_p)$ then the low-fecundity optimal resource stock, $\underline{\hat{x}}_t$, converges to $\underline{\hat{x}}$ from above (see dotted blue trajectories from figs. 3a, 3b and 3d to 3i) so that $\underline{\hat{h}}(x) > \pi\tilde{f}(x)$ for $x \in (\underline{\hat{x}}, x_p)$. But we know that (x_{2t}, h_{2t}) leads to $x_{2\tau} = x_p$ and must have $h_{2t} < \pi\tilde{f}(x_{2t})$ (follows from $\dot{x}_{2t} > 0$). Therefore, $\underline{\hat{h}}(x_{2t}) > h_{2t}$ and it follows that (x_{2t}, h_{2t}) is austere.

3. There are two candidate optimal trajectories and it remains to be determined when (x_{1t}, h_{1t}) is optimal and when (x_{2t}, h_{2t}) is optimal.

   For any trajectory, (x_t, h_t), satisfying (8) and (10), take the ratio of (8) and (10) to get an expression representing the slope of the policy function analogue of (x_t, h_t):

   $$\frac{dh}{dx} = \frac{dh/dt}{dx/dt} = \frac{1}{\sigma(h)}\frac{h[A\tilde{f}'(x) - \rho]}{A\tilde{f}(x) - h}. \tag{A.4}$$

   Now, totally differentiate $\underline{\mathcal{H}}(x, h, \lambda)$ where $\lambda = u'(h)$ and h is a policy function satisfying (8) and (10):

$$\frac{\partial \underline{\mathcal{H}}}{\partial x} + \frac{\partial \underline{\mathcal{H}}}{\partial h}\frac{dh}{dx} + \frac{\partial \underline{\mathcal{H}}}{\partial \lambda}\frac{d\lambda}{dh}\frac{dh}{dx} = u'(h)\pi \tilde{f}'(x) + [u'(h) - \lambda]\frac{dh}{dx} + [\pi \tilde{f}(x) - h]u''(h)\frac{dh}{dx}$$
$$= u'(h)\pi \tilde{f}'(x) + u'(h)[\rho - \pi \tilde{f}'(x)]$$
$$= u'(h)\rho > 0. \tag{A.5}$$

Austerity of $h_2(x)$ implies $u'(h_2(x)) > u'(h_1(x))$ since u is strictly concave. Therefore, (A.5) evaluated at $h_2(x)$ is always greater than when it is evaluated at $h_1(x)$ so that (5) evaluated at $h_2(x)$ is steeper than when evaluated at $h_1(x)$. Together with the HJB equation, this implies that for $x > 0$, value functions $V_1(x)$ and $V_2(x)$ cross at most once. If there is a non-zero crossing point, call it x'_p; otherwise set $x'_p = 0$.

The discounted payoff for trajectory (x_{2t}, h_{2t}) is:

$$V_2(x_0) = \int_0^\tau e^{-\rho t} u(h_{2t})dt + e^{-\rho\tau} V^*(x_p). \tag{A.6}$$

Using the principle of optimality, the discounted payoff for trajectory (x_{1t}, h_{1t}) can be written as:

$$V_1(x_0) = \int_0^\tau e^{-\rho t} u(h_1(x_{1t}))dt + e^{-\rho\tau} V_1(x_{1\tau}). \tag{A.7}$$

When (A.7) is greater than (A.6), trajectory (x_{1t}, h_{1t}) is optimal; when (A.7) is less than (A.6), trajectory (x_{2t}, h_{2t}) is optimal.

Consider $x_0 \in (x_p - \varepsilon, x_p)$ for $\varepsilon > 0$. As $\varepsilon \to 0$, it follows that $\tau \to 0$ and $x_{1\tau} \uparrow x_p$ so that the first terms in each of these equations vanishes while the second terms converge to $V^*(x_p)$ and $\underline{V}(x_p)$.

Since $\underline{\hat{h}}(x_p) \leq \tilde{f}(x_p)$,

$$\begin{aligned} V^*(x_p) &\geq \frac{u(\tilde{f}(x_p))}{\rho} && \text{Proposition 1} \\ &\geq \frac{u(\underline{\hat{h}}(x_p))}{\rho} && \underline{\hat{h}}(x_p) \leq \tilde{f}(x_p) \\ &> \int_0^\infty e^{-\rho t} u(\underline{\hat{h}}_t) dt \\ &= \underline{V}(x_p). \end{aligned}$$

The third inequality follows because for x_0 close to x_p, trajectory $(\underline{\hat{x}}_t, \underline{\hat{h}}_t)$ has $\dot{x}_t, \dot{h}_t < 0$ and thus $\underline{\hat{h}}(x_p) > \underline{\hat{h}}_t$.

Therefore, for x_0 sufficiently close to x_p, the former is greater than the latter and trajectory $(\hat{x}_{2t}, \hat{h}_{2t})$ is optimal.

Now consider x' such that $\lim_{x \downarrow x'} h_2(x) = \pi \tilde{f}(x')$. Since h_2 is austere and $h_2(x) < \pi \tilde{f}(x)$, it must be that if $x' > 0$ then $x' \geq \underline{\hat{x}}$.

When $x' > 0$, let $h_2' = \pi \tilde{f}(x')$. The current value Hamiltonian evaluated at (x', h_2') and $\lambda = u'(h_2')$ is:

$$\underline{\mathscr{H}}(x', h_2', u'(h_2')) = u(\pi \tilde{f}(x')). \tag{A.8}$$

The current value Hamiltonian evaluated at $(x', \hat{h}_1(x'))$ is:

$$\underline{\mathscr{H}}(x', \hat{h}_1(x'), u'(\hat{h}_1(x'))) = u(\hat{h}_1(x')) + u'(\hat{h}_1(x'))[\pi \tilde{f}(x') - \hat{h}_1(x')]) \tag{A.9}$$

Note that for $h \geq \pi \tilde{f}(x)$, (5) with recruitment $\pi \tilde{f}(\cdot)$ is increasing in h (Lemma 1). Since $h_1(x') \geq \pi \tilde{f}(x')$, it must be the case that at x', (A.8) is no greater than (A.9).

Now recall that at $x = x_p$, $\underline{\mathscr{H}}(x_p, \hat{h}_2^-, u'(\hat{h}_2^-)) > \underline{\mathscr{H}}(x_p, \underline{\hat{h}}(x_p), u'(\underline{\hat{h}}(x_p)))$ (part 2a of proof). When $x' > 0$, we know that $x' \geq \underline{\hat{x}}$ and continuity implies that there exists $x_p' \in [x', x_p)$ such that $\underline{\mathscr{H}}(x_p', \hat{h}_2(x_p'), u(\hat{h}_2(x_p')))$ $= \underline{\mathscr{H}}(x_p', \hat{h}_1(x_p'), u'(\hat{h}_1(x_p')))$. When $x' = 0$, since $V_1(x)$ and $V_2(x)$ only cross at $x = 0$, it must be that $V_2(x) > V_1(x)$ for any $x \in (0, x_p)$; set $x_p' = 0$.

The HJB equation implies that for $x < x_p'$, $V_2(x) < V_1(x)$ and for $x > x_p'$, $V_2(x) > V_1(x)$. Therefore,

$$V_*(x) = \begin{cases} V_1(x) & \text{if } x < x_p' \\ V_2(x) & \text{if } x \geq x_p' \end{cases}$$

and

$$\hat{h}_*(x) = \begin{cases} \hat{h}_1(x) & \text{if } x < x_p' \\ \hat{h}_2(x) & \text{if } x \geq x_p'. \end{cases}$$

□

Proof of Proposition 3. For $x_0 < x_p$, we know from Section 3.2 that trajectory $(\hat{x}_{*t}, \hat{h}_{*t})$ is optimal provided that the continuation value at time τ is $e^{-\rho\tau}V^*(x_p)$. Now consider the unconstrained problem for $x_0 \geq x_p$. From Section 3.1, we know that trajectory $(\hat{x}_t^*, \hat{h}_t^*)$ is optimal when x_t is constrained from falling below x_p.

For $x_0 \geq x_p$, if $\hat{\bar{x}} \geq x_p$ then the notional steady-state is feasible and the constraint is non-binding so that trajectory $(\hat{x}_t^*, \hat{h}_t^*) = (\hat{\bar{x}}_t, \hat{\bar{h}}_t)$ is optimal.

If $\hat{\bar{x}} < x_p$ then the question is whether there is an alternative, unconstrained trajectory, (x_t^a, h_t^a), that satisfies (8) and (10) and attains greater welfare, say $V^a(x_0)$. From the argument in the proof of Proposition 1, we know that if $h^a(\hat{x}_t^*) < \hat{h}_t^*$ then it is suboptimal. Alternatively, if $h^a(\hat{x}_t^*) > \hat{h}_t^*$ then x_t^a and h_t^a fall continuously until at some $T < \tau$, $x_T^a = x_p$ and $h_T^a > \tilde{f}(x_p)$ (see fig. 2b). If $x_t^a = x_p$ and $h_t = \tilde{f}(x_p)$ for all $t > T$ then the transversality condition (A.2) fails and (x_t^a, h_t^a) is suboptimal. If x_t^a instead falls below x_p then (8) and (10) imply that $\dot{x} < 0$ and $\dot{h} < 0$. But in this case the only trajectory satisfying (8), (10) and (9) is $(\underline{\hat{x}}_t, \underline{\hat{h}}_t)$. Consider the value from trajectory $(\hat{x}_t^*, \hat{h}_t^*)$ evaluated at x_0:

$$\begin{aligned}
V^*(x_0) &= \int_0^T e^{-\rho t} u(\hat{h}_t^*) dt + \int_T^\tau e^{-\rho t} u(\hat{h}_t^*) dt + \int_\tau^\infty e^{-\rho t} u(\tilde{f}(x_p)) dt \\
&> \int_0^T e^{-\rho t} u(h_t^a) dt + \int_T^\tau e^{-\rho t} u(\tilde{f}(x_p)) dt + \int_\tau^\infty e^{-\rho t} u(\tilde{f}(x_p)) dt \\
&> \int_0^T e^{-\rho t} u(h_t^a) dt + \int_T^\tau e^{-\rho t} u(\underline{\hat{h}}_t) dt + \int_\tau^\infty e^{-\rho t} u(\underline{\hat{h}}_t) dt \\
&= V^a(x_0).
\end{aligned}$$

The first inequality follows from the fact that $(\hat{x}_t^*, \hat{h}_t^*)$ is constrained optimal (Section 3.1) and h_t^a for $t \in [0, T)$ and $\tilde{f}(x_p)$ for $t \in [T, \tau)$ are feasible constrained harvests. The second inequality holds whenever $\underline{\hat{h}}(x_p) \leq \tilde{f}(x_p)$ because $\dot{\underline{\hat{h}}}_t < 0$ for $t > T$. Therefore $(\hat{x}_t^*, \hat{h}_t^*)$ is optimal for the unconstrained problem. Since $(\hat{x}_t^*, \hat{h}_t^*)$ is unconstrained optimal, $(\hat{x}_{*t}, \hat{h}_{*t})$ is optimal when $x_0 < x_p$. □

Proof of Proposition 4. As in the case without hysteresis, the problem will be divided between the high-fecundity problem where $x_0 \geq x_p$, $s = 1$ and recruitment is given by $\tilde{f}(x)$ and the low-fecundity problem where $x_0 < x_p^\hbar$, $s = 0$ and recruitment is given by $\pi \tilde{f}(x)$.

The analysis for the high-fecundity problem with hysteresis is identical to the analysis without hysteresis and the optimal solution has trajectory $(\hat{x}_t^{\hbar *}, \hat{h}_t^{\hbar *}) = (\hat{x}_t^*, \hat{h}_t^*)$ for $t \geq 0$ and value $V^{\hbar *}(x) = V^*(x)$.

For the low-fecundity problem with hysteresis, the analysis of trajectories that converge to the low notional steady-state is identical to the model without hysteresis so that $(x_{1t}^\hbar, h_{1t}^\hbar) = (x_{1t}, h_{1t})$ and $V_1^\hbar(x) = V_1(x) = \underline{V}(x)$. The analysis for the trajectory that transitions to high-fecundity recruitment is slightly different and now occurs at some time T such that $x_{2T}^\hbar = x_p^\hbar > x_p$. In this case, the transversality condition is now:

$$\lim_{t \to T} \underline{\mathcal{H}}(x_t, h_t, \lambda_t) = \rho V^{\hbar *}(x_p^\hbar). \tag{A.10}$$

The proof is identical; let the optimal trajectory be given by $(x_{2t}^\hbar, h_{2t}^\hbar)$ with value $V_2^\hbar(x)$.

Proof of the optimality of the composite trajectory is the same as for Proposition 3.

Note that it must be the case that, where their domains overlap, $V_2(x) > V_2^{\not h}(x)$ since without hysteresis, consumption path $\hat{h}_{2t}^{\not h}$ is feasible but $\hat{h}_{2t}$ is uniquely optimal. Clearly, $V_1(x) = V_1^{\not h}(x)$. Together, this implies that $x_p' < x_p^{\not h\prime}$. $\square$

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