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Risk and evolution★

Ted To[1,2]
[1]Department of Economics, University of Warwick, Warwick, Coventry CV4 7AL, UK
[2]Department of Economics, University of St. Andrews, St. Andrews, Fife KY16 9AL, UK
(e-mail: tct1@st-and.ac.uk)

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Summary. I examine a Knightian (1921) model of risk using a general equilibrium model of investment and trade. A population of agents with various preference types can choose between a safe production technology and a risky production technology. In addition, the distribution of types of agents changes through a standard evolutionary dynamic. For a given population distribution, the equilibrium is in general inefficient, however, by allowing the population distribution to change in response to market generated rewards, the population will converge to one where the equilibrium is efficient and where the population as a whole behaves as if all agents were risk neutral.

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1 Introduction

I examine a Knightian (1921) model of risk using a general equilibrium model of investment and trade. Agents of various preference types can choose between a safe and a risky production technology. Risk in this model is uninsurable for reasons of moral hazard.[1] Risks are independently and identically distributed and once the outcomes of these production choices are

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[1] For example, if the success of investments depend on unobservable effort then there cannot be full insurance. Alternatively, Kihlstrom and Laffont (1979) assume institutional restrictions against risk trading.

realized, agents trade in a competitive exchange market. Finally, preference types which induce successful behavior become more numerous. My model ties together three distinct but related literatures: Blume and Easley (1992), Kihlstrom and Laffont (1979) and papers on the evolution of attitudes towards risk (Robson, 1996, 1997).

Kihlstrom and Laffont (1979) model entrepreneurial self-selection through agents' risk preferences. They, construct a general equilibrium model where agents with differing attitudes towards risk can choose between working for a certain wage or operating a firm for an uncertain return. One of their main results is that, in general, the equilibrium is Pareto inefficient. This is because with non-risk-loving individuals, there is, in general, an inefficiently low level of aggregate investment. However, if risk preferences adjust in response to market rewards, in the long run, investments need not be inefficient.

Blume and Easley (1992) model asset pricing with a wealth accumulation process that has parallels to models of evolution. They find that given a finite set of agents, those that eventually hold almost all of the wealth are risk-averse. One might conjecture that long run risk-aversion is a result of their finite population – if an agent did not own a share of every asset, then with probability one, that agent would hold no wealth in finite time. However, it can be seen that this difference still obtains even when there are many agents. In particular, since the returns earned by agents holding identical portfolios are perfectly correlated, any preference type which did not own a share of every asset would hold no wealth in finite time. When risks are independent, risk-aversion no longer results.

Finally, I depart from the literature on evolution and risk attitudes in that the relative payoffs are determined endogenously through a competitive pricing mechanism. Robson (1996, 1997), for example, examines decision theoretic models where fitness payoffs in each period are fixed (i.e., the lotteries between which agents are choosing are not priced). This generically precludes long run populations where more than one action is chosen.

I assume that the distribution of types of agents to changes in response to market generated rewards – this occurs through the standard replicator dynamic. In particular, preference types that do well, increase in relative frequency.[2] Dekel and Scotchmer (1992) provide two possible motivations for the use of replicator dynamics. One is to argue that higher wealth corresponds to more children and children inherit the preferences of their parents. The second is to argue that the replicator dynamic represents a reduced form learning and imitation model with bounded rationality. This argument has theoretical underpinnings which include Börgers and Sarin (1993), Hopkins (1995) and Schlag (1994) who demonstrate the close relationship between various models of learning and imitation and the standard replicator dynamic. Finally, as I demonstrate in Section 7, the replicator dynamic is

[2] See Bergstrom (1995), Güth and Yaari (1992), Rogers (1994), and Waldman (1994) for other examples which study evolution over preferences.

formally identical to the Blume and Easley (1992) notion that the effective distribution of types changes as a result of wealth accumulation.

The main results follow. For a given population distribution, the equilibrium is in general inefficient. This results because in equilibrium there is an inefficient allocation of risk. Unlike Kihlstrom and Laffont, the equilibrium is inefficient not only because there can be too little investment but because when a portion of the population is made up of risk-lovers, there can also be too much investment. However, by allowing the population distribution to change in response to market generated relative rewards, the population will converge to one where the equilibrium is efficient and where the population as a whole behaves as if all agents were risk neutral.

2 The model

In each period, individuals make investment decisions. After the outcomes of their investments have been realized, they participate in a competitive exchange market. Investment decisions and trades are chosen to maximize utility, however, preferences change over time and agents that do well earn greater representation in succeeding generations.

There are two goods, 1 and 2, denoted by index i. Assume that there is a continuum of types of agents, $\bar{A} = [\underline{\alpha}, \bar{\alpha}]$ where $\underline{\alpha} < 1 < \bar{\alpha}$. Let μ be an atomless probability measure with support $\bar{A}$.[3] Probability measure μ describes the distribution of agents of each type.

All individuals have an endowment (e.g., time, money, etc.) which can produce a single unit of good 1 with certainty. It can alternatively be invested to produce a single unit of good 2 with probability σ and nothing with probability $(1 - \sigma)$.[4] Assume the realization of each individual's investment is independent of the realizations of all other investments.

This model can be interpreted as one where each agent has an endowment of labor which can be 'invested' in a risky activity (hunting or operating a firm) or a safe activity (gathering or working for a wage). Or alternatively as one where agents are initially endowed with a good which can be transformed, using a risky technology, into another good. For example, money held in a bank account provides a sure return – money can also be invested in a risky project which provides an uncertain monetary return.

An individual of type $\alpha \in \bar{A}$ has expected utility function $u_\alpha(\cdot,\cdot) = f(\cdot,\cdot)^\alpha$, where $f : \mathbb{R}^2_+ \to \mathbb{R}_+$ is homogeneous of degree 1. Assume that $f(x_1, x_2) = 0$ only if $x_1 = 0$ or $x_2 = 0$, ensuring positive consumption of both goods in equilibrium. Since u is a monotonic transformation of f, the

[3] When μ has atoms, results are analogous to those found with atomless μ. When distributions have atoms, the notation becomes more complicated because types that are indifferent may have mass and one would have to keep track of the mass of those that choose each action. One such example is considered in Section 6.

[4] One would not expect that the results would change if investments decisions are divisible. For example, it is easy to show with a finite type example with divisible investments that the results extend.

demand of every type is identical and can be expressed as $x_i^*(p,m)$, where $p = p_1/p_2$ and m is wealth. Assume that goods 1 and 2 are gross substitutes (i.e., $\partial x_2^*(p,m)/\partial p \geq 0$ and $\partial x_1^*(p,m)/\partial p < 0$).

The value $f(x_1, x_2)$ is the evolutionary fitness earned by an agent consuming x_1 and x_2, and f^α is the utility a type α agent derives from fitness f. Constant relative risk aversion provides a convenient means for characterizing agents with different attitudes towards risk. Homogeneity of fitness has the advantage that an agent's risk behavior is due solely to her utility function. Furthermore, with homogeneity, fitness becomes a multiple of wealth or income and thus changes in the population distribution are driven by relative wealth.

I assume the population evolves following a natural generalization of the discrete time replicator dynamic. Let $f_t(\alpha)$ be the average fitness achieved by type α agents. Using t to denote time, the population evolves according to:

$$\mu_{t+1}(A) = \frac{\int_A f_t(\alpha)\, d\mu_t(\alpha)}{\int_{\underline{\alpha}}^{\bar{\alpha}} f_t(\alpha)\, d\mu_t(\alpha)} \tag{1}$$

for all measurable $A \subset \bar{A}$. If a population evolves according to (1), types which achieve fitness greater than the population average earn greater re- presentation next period.

3 Market equilibrium

As is standard in multi-stage models, I work backwards and first solve for the competitive equilibrium given investment decisions. Let $e_1, e_2 : \bar{A} \to \{0,1\}$ give each type's investment decision with $e_1(\cdot) + e_2(\cdot) = 1$ where $e_2(\alpha) = 0$ if α type individuals do not invest and $e_2(\alpha) = 1$ if they do. I assume that all individuals of the same type choose the same investment levels and that $e_2(\cdot)$ is measurable – this will be true in equilibrium.

Let good 2 be the numeraire and $p = p_1/p_2$ be the price ratio. Given agents' investment decisions, $e_1(\cdot)$ and $e_2(\cdot)$, a competitive exchange equilibrium is a triplet $(x_1^*(\cdot,\cdot), x_2^*(\cdot,\cdot), p^e)$ such that i) $(x_1^*(p,m), x_2^*(p,m))$ maximizes utility given price, p, and wealth, m, ii) p^e is such that aggregate demand for each good is equal to the aggregate 'endowment.'

Since the function underlying utility is homogeneous, the utility maximizing demands are $x_i^*(p,m) = x_i^*(p)m$. The law of large numbers implies that the average wealth of type α agents is $pe_1(\alpha) + \sigma e_2(\alpha)$ so that aggregate wealth is $\int_{\underline{\alpha}}^{\bar{\alpha}} (pe_1(\alpha) + \sigma e_2(\alpha)) d\mu(\alpha)$. Similarly, the aggregate supply of goods 1 and 2 are $\int_{\underline{\alpha}}^{\bar{\alpha}} e_1(\alpha) d\mu(\alpha)$ and $\sigma \int_{\underline{\alpha}}^{\bar{\alpha}} e_2(\alpha) d\mu(\alpha)$. Therefore the market clearing conditions for goods 1 and 2 are:

$$\begin{aligned} x_1^*(p^e) \int_{\underline{\alpha}}^{\bar{\alpha}} (p^e e_1(\alpha) + \sigma e_2(\alpha))\, d\mu(\alpha) &= \int_{\underline{\alpha}}^{\bar{\alpha}} e_1(\alpha)\, d\mu(\alpha) \\ x_2^*(p^e) \int_{\underline{\alpha}}^{\bar{\alpha}} (p^e e_1(\alpha) + \sigma e_2(\alpha))\, d\mu(\alpha) &= \sigma \int_{\underline{\alpha}}^{\bar{\alpha}} e_2(\alpha) d\mu(\alpha)\ . \end{aligned} \tag{2}$$

Gross substitutability implies uniqueness of the competitive exchange equilibrium.

4 Investment equilibrium

Given the anticipated outcome of the second stage exchange market, each individual makes her investment decision to maximize expected utility. Expectations about the price in the second stage exchange market are consistent in that given the aggregate equilibrium investment levels, these expectations are realized.

To get agents' expected utilities, I first substitute their demands into the fitness function to get an indirect fitness function, $g(p,m) = f(x_1^*(p,m), x_2^*(p,m))$. Since the fitness function is homogeneous, the indirect fitness function has the property that it can be expressed as $g(p,m) = g(p)m$. Hence type α agents have indirect utility function $v_\alpha(p,m) = [g(p)m]^\alpha$. Using the indirect utility function, a type α individual solves:

$$\begin{aligned} \max_{e_1(\alpha),e_2(\alpha)} \quad & g(p)^\alpha[\sigma(pe_1(\alpha)+e_2(\alpha))^\alpha + (1-\sigma)(pe_1(\alpha))^\alpha] \\ & \text{subject to: } e_1(\cdot)+e_2(\cdot)=1, \\ & e_1(\cdot), e_2(\cdot) \in \{0,1\}. \end{aligned} \tag{3}$$

A Nash equilibrium in investments is given by a triplet $(e_1^*(\cdot\,,\cdot), e_2^*(\cdot\,,\cdot), p)$ if i) given $p(e_1^*(\alpha,p), e_2^*(\alpha,p))$ solves (3), for any α and ii) $p = p^e$ where p^e satisfies (2) given $(e_1(\cdot), e_2(\cdot)) = (e_1^*(\cdot\,,p), e_2^*(\cdot\,,p))$. In other words, $e_1^*(\cdot\,,p)$, $e_2^*(\cdot\,,p)$ maximizes utility when the price is p and p clears the market for these investment strategies.

Before characterizing the equilibria of the investment stage, define $n^{RL} = \mu[1,\bar{\alpha}]$ (i.e., n^{RL} is the mass of the risk-loving agents).

Theorem 1 *Equilibria of the investment stage*, $(e_1^*(\cdot,\cdot), e_2^*(\cdot,\cdot), p)$, *exist and satisfy*:

i) $p \in (\underline{p}, \bar{p})$ *where* $\underline{p} = \sigma^{1/\underline{\alpha}}$ *and* $\bar{p} = \sigma^{1/\bar{\alpha}}$,

ii) $p\left\{\begin{matrix} < \\ = \\ > \end{matrix}\right\}\sigma$ *as* $n^{RL}\left\{\begin{matrix} < \\ = \\ > \end{matrix}\right\}x_2^*(\sigma)$ *and*

iii) *if* $p < \sigma^{1/\alpha}$ *then* $e_2^*(\alpha,p) = 1$, *if* $p > \sigma^{1/\alpha}$ *then* $e_2^*(\alpha,p) = 0$.

Furthermore, the equilibrium price and the aggregate level of investment are unique.

Proof. I first show that any equilibrium of the investment stage must satisfy iii) and then show that given iii) the investment stage has an equilibrium with a unique price and aggregate investments. Finally, I prove i) and ii).

iii) each agent's expected utilities from investing and not investing is $g(p)^\alpha\sigma$ and $g(p)^\alpha p^\alpha$. Let $p = \sigma^{1/\alpha}$ be the price at which an individual of type α is indifferent between investing and not investing. Hence if $p < \sigma^{1/\alpha}$ then $e_2^*(\alpha,p) = 1$ and if $p > \sigma^{1/\alpha}$ then $e_2^*(\alpha,p) = 0$.

Now, to prove the existence of an equilibrium and the uniqueness of price and aggregate investments, note that *any* equilibrium of the investment stage must satisfy iii). It will be useful here to define the critical value $\alpha^*(p) = \ln \sigma / \ln p$. Given p, any agent of type $\alpha < \alpha^*(p)$ prefers not to invest and any agent of type $\alpha > \alpha^*(p)$ prefers to invest (i.e., $\alpha^*(p)$ solves $p = \sigma^{1/\alpha}$). For a given population distribution, this implies that the mass of agents who invest, $\mu[\alpha^*(p), \bar{\alpha}]$, is decreasing in p. Hence the aggregate investment function, $I(p) = \int_{\underline{\alpha}}^{\bar{\alpha}} e_2^*(\alpha, p) d\mu(\alpha) = \mu[\alpha^*(p), \bar{\alpha}]$, must be decreasing and continuous in p. Next, from the (2) it can be seen that for a given aggregate investment level, $I = \int_{\underline{\alpha}}^{\bar{\alpha}} e_2(\alpha) d\mu(\alpha)$, the implicit function $p^e(I)$ must satisfy: $x_2^*(p^e)p^e/((1 - x_2^*(p^e))\sigma) = I/(1 - I)$. Since $\partial x_2^*/\partial p^e \geq 0$, $p^e(I)$ is continuous and increasing in aggregate investment. Thus the composite function $I \circ p^e : [0, 1] \to [0, 1]$ is continuous and decreasing and has a unique fixed point. Thus an equilibrium exists and price and aggregate investment are unique.

i) Note that $\bar{p} = \sigma^{1/\bar{\alpha}}$ is the price at which agents of type $\bar{\alpha}$ are indifferent between investing and not investing. If $p \geq \bar{p}$ then almost all agents strictly prefer to invest nothing and hence everyone gets zero utility. All agents would thus prefer to invest since they could then achieve infinite expected utility. Thus an upper-bound on the equilibrium price is $\bar{p}$. Similarly, the equilibrium price is bounded below by $\underline{p} = \sigma^{1/\underline{\alpha}}$. Since $\underline{p} < \bar{p}$, $p \in (\underline{p}, \bar{p})$.

ii) Note from (2) and the optimal investment strategies, iii), that when $p = \sigma$, $n^{RL} = x_2^*(\sigma)$. W.l.o.g. suppose that $p < \sigma$. Using the equilibrium investment strategies, solve the market clearing condition for good 2 at prices p and σ for $x_2^*(p)$ and $x_2^*(\sigma)$.

$$x_2^*(p) = \frac{\sigma\left(\int_{\underline{\alpha}}^{1} e_2^*(\alpha, p)\, d\mu + n^{RL}\right)}{\int_{\underline{\alpha}}^{1} (pe_1^*(\alpha, p) + \sigma e_2^*(\alpha, p))\, d\mu + \sigma n^{RL}} \tag{4}$$

$$x_2^*(\sigma) = \frac{\sigma x_2^*(\sigma)}{\sigma} \tag{5}$$

Suppose that $n^{RL} \geq x_2^*(\sigma)$. This implies that the top of the right hand side of (4) is strictly greater than the top of the right hand side of (5) since for $\alpha \in (\alpha^*(p), 1]$, $e_2^*(\alpha, p) = 1$. Furthermore, since $p < \sigma$, the bottom of the right hand side of (4) is strictly less than σ, the bottom of the right hand side of (5). Hence, the right hand side of (4) is greater than the right hand side of (5). Since $\partial x_2^*/\partial p \geq 0$ it must be that $p > \sigma$, a contradiction. Therefore $n^{RL} < x_2^*(\sigma)$. The proof of the converse is similar. ■

Parts i) and ii) characterize the equilibrium prices. First, as the likelihood of a successful investment decreases, the band in which the equilibrium price ratio must reside moves downwards. That is, in order to induce investment the reward for investing must be sufficiently high, relative to that for not investing. Second, the value of good 1 is low when there are relatively few risk-loving types and high when there are relatively few risk-averse types. Finally, $\bar{p} \leq 1$ so that the return from a successful investment is always greater than that from not investing, however, the *expected* return may be greater or less than that of not investing, depending on the relative numbers of agents participating in each activity.

Part iii) characterizes individual investment decisions given price – the risk-averse invest if the expected return from investing is relatively high and the risk-loving don't invest only if the expected return from investing is low. In general, risk-loving types are more likely to invest than risk-averse types. Finally, although equilibrium investment strategies are not unique (i.e., one measure-zero type is indifferent between investing and not investing), price and aggregate investment are unique and hence the equilibrium is almost unique.

Finally, given p, the functions $e_1^*(\cdot\,,p)$ and $e_2^*(\cdot\,,p)$ are discontinuous at most at one α and so are measurable. Letting $e_i(\cdot) = e_i^*(\cdot,p)$ justifies the assumption of Section 3.

At this point it is useful to refer to the Kihlstrom and Laffont (1979) welfare result which says that in an economy which has institutional constraints on risk trading, efficiency is achieved only if all investors are risk neutral. In the current model, depending on the distribution of risk attitudes, aggregate investment will in general be inappropriate and as a result, the allocation of risk will be inefficient.

5 The evolution of risk attitudes

So far, the above analysis has been purely static. Now suppose that the population evolves in response to market rewards, according to the discrete time replicator dynamic. Employing the properties of the investment and exchange equilibrium and the law of large numbers, type α average fitness in period t is $f_t(\alpha) = g(p_t)(p_t e_1^*(\alpha,p_t) + \sigma e_2^*(\alpha,p_t))$. Hence the evolutionary population dynamics equation (1) becomes:

$$\mu_{t+1}(A) = \frac{\int_A (p_t e_1^*(\alpha,p_t) + \sigma e_2^*(\alpha,p_t))\, d\mu_t(\alpha)}{\int_{\underline{\alpha}}^{\bar{\alpha}} (p_t e_1^*(\alpha,p_t) + \sigma e_2^*(\alpha,p_t))\, d\mu_t(\alpha)} \tag{6}$$

for all measurable $A \subset \bar{A}$ and where $g(p_t)$ has been factored out and cancelled from both the numerator and denominator. Since $e_1^*(\alpha,p_t), e_2^*(\alpha,p_t) \geq 0$, $e_1^*(\alpha,p_t) + e_2^*(\alpha,p_t) = 1$ and p_t is bounded, it is clear that if μ_t is an atomless probability measure with support $\bar{A}$, then μ_{t+1} is also an atomless probability measure with support $\bar{A}$. Notice that the propagation of each type depends only on the average value of its 'endowment' or wealth – the relative frequency of a type increases if its wealth is greater than the average population wealth.

A population distribution, μ_t with support $\bar{A}$ is **stationary** if and only if $\mu_{t+1}(A) = \mu_t(A)$ for all measurable $A \subset \bar{A}$, where μ_{t+1} follows (6). Let $\mathscr{M}$ be the family of such probability measures that are atomless. Define p_μ to be the equilibrium price, given μ.

Theorem 2 *An atomless probability measure, μ with support $\bar{A}$ is an element of $\mathscr{M}$ if and only if $p_\mu = \sigma$ or equivalently if and only if $\mu[1,\bar{\alpha}] = x_2^*(\sigma)$. Furthermore, if μ_1 is atomless, and μ_t follows* (6) *then $\mu_t \to \mu$ for some $\mu \in \mathscr{M}$.*

Proof. Let μ_1 be an atomless probability measure with support $\bar{A}$ and let μ_t follow (6). Define p_t to be the equilibrium price that obtains if the population distribution is μ_t.

The strategy of the proof is as follows. 1) Characterize the set of stationary population distributions – every element of this set will have the property that the mass of the risk-loving types is equal to $x_2^*(\sigma)$. 2) Show that the mass of the risk-loving types, n_t^{RL}, converges. 3) Show that $p_t \to \sigma$ and $n_t^{RL} \to x_2^*(\sigma)$. 4) Show that $\mu_t \to \mu \in \mathscr{M}$.

1) Suppose μ is stationary. By examining (6) it must be that either $p_\mu = \sigma$ or else for some i, $e_i(\alpha, p_\mu) = 1$ for almost all α. Since not all types choose the same action (from Theorem 1), it must be that $p_\mu = \sigma$. Conversely, if $p_\mu = \sigma$ then by (6), μ must be stationary.

Using Theorem 1 and examining the market clearing condition for good 2 (2), it can be seen that if μ is such that $p_\mu = \sigma$ then $\mu[1, \bar{\alpha}] = x_2^*(\sigma)$. Conversely, suppose that $\mu[1, \bar{\alpha}] = x_2^*(\sigma)$ but $p_\mu \neq \sigma$. Without loss of generality assume that $p_\mu < \sigma$. Then:

$$x_2^*(p_\mu) \int_{\underline{\alpha}}^{\bar{\alpha}} (p_\mu e_1^*(\alpha, p_\mu) + \sigma e_2^*(\alpha, p_\mu))\, d\mu(\alpha) < \sigma\mu[1, \bar{\alpha}] \tag{7}$$

$$< \sigma\mu[\alpha^*(p_\mu), \bar{\alpha}] \tag{8}$$

$$= \sigma \int_{\underline{\alpha}}^{\bar{\alpha}} e_2^*(\alpha, p_\mu)\, d\mu(\alpha) \tag{9}$$

(7) follows by assumption because $x_2^*(p_\mu) < x_2^*(\sigma) = \mu[1, \bar{\alpha}]$ and because $\int_{\underline{\alpha}}^{\bar{\alpha}} (p_\mu e_1^*(\alpha, p_\mu) + \sigma e_2^*(\alpha, p_\mu)) d\mu(\alpha) < \sigma$, (8) follows because when $p_\mu < \sigma$, $\alpha^*(p_\mu)$ as defined in the proof of Theorem 1 is strictly less than 1, (9) also follows from Theorem 1 because for $\alpha > \alpha^*(p_\mu)$, $e_2^*(\alpha, p_\mu) = 1$ and for $\alpha < \alpha^*(p_\mu)$, $e_2^*(\alpha, p_\mu) = 0$. This contradicts the market clearing conditions (i.e., the left hand side of (7) must equal (9)) and therefore $p_\mu = \sigma$.

W.l.o.g., assume for the remainder of the proof that μ_1 is such that $p_1 < \sigma$.

2) First I show that if μ_1 is such that $p_1 < (>) \sigma$ then n_t^{RL} is an increasing (decreasing) sequence bounded above (below) by $x_2^*(\sigma)$. Given the equilibrium investment strategies,

$$n_{t+1}^{RL} = \mu_{t+1}[1, \bar{\alpha}] \tag{10}$$

$$= \frac{\sigma n_t^{RL}}{\int_{\underline{\alpha}}^1 (p_t e_1^*(\alpha, p_t) + \sigma e_2^*(\alpha, p_t))\, d\mu_t + \sigma n_t^{RL}} \tag{11}$$

$$< \frac{\sigma\left(\int_{\underline{\alpha}}^1 e_2^*(\alpha, p_t)\, d\mu_t + n_t^{RL}\right)}{\int_{\underline{\alpha}}^1 (p_t e_1^*(\alpha, p_t) + \sigma e_2^*(\alpha, p_t))\, d\mu_t + \sigma n_t^{RL}} \tag{12}$$

$$= x_2^*(p_t) \tag{13}$$

$$\leq x_2^*(\sigma) \tag{14}$$

(11) follows from the definition of the population dynamic (6) and Theorem 1, (12) comes from adding the term $\sigma \int_{\underline{\alpha}}^1 e_2(\alpha)\, d\mu_t(\alpha)$ to the numerator of (11),

(13) follows from (4), and (14) follows from the fact that $x_2^*(\cdot)$ is increasing. Hence $n_{t+1}^{RL} < x_2^*(\sigma)$. Finally, (11) and the fact that $p_t < \sigma$ implies that $n_t^{RL} < n_{t+1}^{RL}$. Therefore n_t^{RL} is an increasing sequence bounded above by $x_2^*(\sigma)$. Since n_t^{RL} is monotonic and bounded, it converges.

3) By examining (11), it can be seen that since $e_1^*(\alpha, p_t) = 1$ for all $\alpha < \alpha^*(p_t)$, n_t^{RL} converges only if either $p_t \to \sigma$ or $n_t^{RL} \to 1$. Since part 2) of the proof implies that $n_t^{RL} < x_2^*(\sigma)$ for all t and that $x_2^*(\sigma) < 1$ (i.e., consumption of both goods must be positive in equilibrium, so $x_2^*(\sigma) < 1$), it must be that $p_t \to \sigma$. Noticing that the market clearing condition for good 2 (equation (2)) is continuous in price, if $p_t \to \sigma$ then $n_t^{RL} \to x_2^*(\sigma)$.

4) The strategy here will be as follows. First show that for any interval $\mathscr{I} \subset [1, \bar{\alpha}]$, $\mu_t(\mathscr{I})$ converges. Then for any $a < 1$ and any interval $\mathscr{I} \subset [\underline{\alpha}, a]$, show that $\mu_t(\mathscr{I})$ converges. Next, for any measurable $A \subset \bar{A}$, show that $\mu_t(A)$ converges. Call that convergent point $\mu(A)$. It is clear that μ so defined is a probability measure with support $\bar{A}$ and that $\mu_t \xrightarrow{v} \mu$. Lastly, show that μ has support $\bar{\alpha}$. Since every μ_t is atomless, μ must be atomless and $\mu \in \mathscr{M}$.

Let $f_t(\alpha) = p_t e_1^*(\alpha, p_t) + \sigma e_2^*(\alpha, p_t)$ and $c_t = \int_{\underline{\alpha}}^{\bar{\alpha}} f_t(\alpha)\, d\mu_t(\alpha)$. Redefine (6) as a density function: $d\mu_{t+1}(\alpha) = (f_t(\alpha)/c_t)\, d\mu_t(\alpha)$. Since $p_{t+n} < \sigma$, $f_{t+n}(\alpha) = \sigma$ for any $n \geq 0$ and any $\alpha > 1$. Hence,

$$\begin{aligned}\mu_{t+n}[1, \bar{\alpha}] &= \int_1^{\bar{\alpha}} \frac{f_{t+n-1}(\alpha) f_{t+n-2}(\alpha) \cdots f_t(\alpha)}{c_{t+n-1} c_{t+n-2} \cdots c_t}\, d\mu_t(\alpha) \\ &= \frac{\sigma^n}{c_{t+n-1} c_{t+n-2} \cdots c_t}\, \mu_t[1, \bar{\alpha}] \ .\end{aligned}$$

But $\mu_{t+n}[1, \bar{\alpha}] \to x_2^*(\sigma)$ as $n \to \infty$. Since $\mu_t[1, \bar{\alpha}] > 0$,

$$\lim_{n \to \infty} \frac{\sigma^n}{c_{t+n-1} c_{t+n-2} \cdots c_t} = \frac{x_2^*(\sigma)}{\mu_t[1, \bar{\alpha}]} \ .$$

Therefore for any interval $\mathscr{I} \subset [1, \bar{\alpha}]$,

$$\mu_{t+n}(\mathscr{I}) = \frac{\sigma^n}{c_{t+n-1} c_{t+n-2} \cdots c_t} \mu_t(\mathscr{I})$$

also converges as $n \to \infty$.

Now consider any $a < 1$ and any interval $\mathscr{I} \subset [\underline{\alpha}, a] \subset [\underline{\alpha}, 1)$. Since $\alpha^*(p_t) \to 1$, then for sufficiently large t, $e_1^*(\alpha, p_t) = 1$ for all $\alpha \in \mathscr{I}$, implying that $f_t(\alpha) = p_t$ or

$$\begin{aligned}\mu_{t+n}(\mathscr{I}) &= \int_{\mathscr{I}} \frac{p_{t+n-1} p_{t+n-2} \cdots p_t}{c_{t+n-1} c_{t+n-2} \cdots c_t}\, d\mu_t(\alpha) \\ &= \frac{p_{t+n-1} p_{t+n-2} \cdots p_t}{c_{t+n-1} c_{t+n-2} \cdots c_t}\, \mu_t(\mathscr{I}) \ .\end{aligned}$$

Consider the sequence:

$$\left\{ \frac{p_{t+n-1} p_{t+n-2} \cdots p_t}{c_{t+n-1} c_{t+n-2} \cdots c_t} \right\}_{n=0}^{\infty} \ .$$

This is a decreasing sequence since for every $t' \geq t$, $p_{t'}/c_{t'} < 1$. Since it is bounded below by zero, it converges so that for any $a < 1$ and any $\mathscr{I} \subset [\underline{\alpha}, a]$, $\mu_{t+n}(\mathscr{I})$ converges as $n \to \infty$.

Next, let $A \subset \bar{A}$ be a measurable set. Define a sequence of real numbers $\{z_m\}$ such that $z_1 = \underline{\alpha}$, $z_{m+1} > z_m$ and $z_m \to 1$. It clearly follows that the intervals $Z_1 = [z_1, z_2]$ and $Z_m = (z_m, z_{m+1}]$, $m \geq 2$, are disjoint and $\bigcup_{m=1}^{\infty} Z_m = [\underline{\alpha}, 1)$. For each m, $Z_m \subset [\underline{\alpha}, a]$ for some $a < 1$ and hence $\mu_t(Z_m)$ converges. By the reasoning in the previous paragraph, $\mu_t(A \cap Z_m)$ also converges. Similarly, for $\mathscr{I} \in [1, \bar{\alpha}]$, $\mu_t(A \cap \mathscr{I})$ converges. Therefore,

$$\begin{aligned}\mu_t(A) &= \mu_t\left(\left(\bigcup_{m=1}^{\infty} Z_m \cup [1, \bar{\alpha}]\right) \cap A\right) \\ &= \sum_{m=1}^{\infty} \mu_t(Z_m \cap A) + \mu_t([1, \bar{\alpha}] \cap A)\end{aligned}$$

converges. For all measurable $A \subset \bar{A}$ define $\mu(A)$ to be this limit. It is easy to see that μ so defined is a probability measure with support $\bar{A}$. Hence $\mu_t \xrightarrow{v} \mu$. Since every μ_t is atomless, vague convergence implies that μ is atomless (Chung, 1974).

Probability measure μ has support $\bar{A}$ if and only if for any $\alpha \in \bar{A}$ and any $\epsilon > 0$, $\mu(N(\alpha, \epsilon)) > 0$. First, consider $\alpha \geq 1$. It is clear from above that $\mu_t(N(\alpha, \epsilon) \cap [1, \bar{\alpha}]) > 0$ is an increasing sequence so $\mu(N(\alpha, \epsilon)) > 0$. Second, for $\alpha < 1$, the decreasing sequence

$$\left\{\frac{p_{t+n-1} p_{t+n-2} \cdots p_t}{c_{t+n-1} c_{t+n-2} \cdots c_t}\right\}_{n=0}^{\infty}$$

is part of

$$\begin{aligned}\mu_{t+n}(N(\alpha, \epsilon) \cap [\underline{\alpha}, a]) &= \int_{N(\alpha,\epsilon)\cap[\underline{\alpha},1)} \frac{p_{t+n-1} p_{t+n-2} \cdots p_t}{c_{t+n-1} c_{t+n-2} \cdots c_t}\, d\mu_t(\alpha) \\ &= \frac{p_{t+n-1} p_{t+n-2} \cdots p_t}{c_{t+n-1} c_{t+n-2} \cdots c_t}\, \mu_t(N(\alpha, \epsilon) \cap [\underline{\alpha}, a])\end{aligned}$$

for some $\alpha < a < 1$ and sufficiently large t. Hence if $\mu(N(\alpha, \epsilon) \cap [\underline{\alpha}, 1)) = 0$ for any $\alpha < 1$ it must be true for every $\alpha < 1$. (I.e., the convergence of any sequence is driven by its tail and the tail of every sequence here is identical.) Since μ is atomless, this would contradict $\mu[\underline{\alpha}, 1) = 1 - n^{RL} > 0$. Therefore μ has support $\bar{A}$.

Since $\mu[1, \bar{\alpha}] = \lim_{t\to\infty} \mu_t[1, \bar{\alpha}] = x_2^*(\sigma)$, $\mu \in \mathscr{M}$. ■

Theorem 2 demonstrates that all stationary distributions have the property that the masses of the risk-averse types and the risk-loving types depend only on the fitness function and the probability of a successful investment. These stationary distributions are such that the risk-averse do not invest and the risk-loving invest. Furthermore, it immediately follows that for any stationary distribution, $\mu \in \mathscr{M}$, there exists an initial distribution, μ_1, such that $\mu_t \xrightarrow{v} \mu$. More importantly, these stationary distributions are stable in the sense that if a mutation occurs, the population will converge to another stationary distribution.

It will also be interesting to consider population distributions with support over some subsets of the following type, $A_1 \subset [\underline{\alpha}, 1]$ and $A_2 \subset [1, \bar{\alpha}]$. It follows from Theorem 2 that with small mutations, no distribution with support A_1 or A_2 can be stable.

Corollary 1 *Let μ^* be an atomless distribution with support $A_1 \subset [\underline{\alpha}, 1)$ or support $A_2 \subset (1, \bar{\alpha}]$. Let $\epsilon > 0$ and μ_1 be an atomless perturbation of μ^* with support $\bar{A}$ and where $\mu_1(A_i) = \mu^*(A_i) - \epsilon$ and $\mu_1(\bar{A} - A_i) = \epsilon$. If μ_1 follows* (6) *then $\mu_t \to \mu \in \mathscr{M}$.*

Finally, given the model of investment and exchange, the distribution over preferences can be examined for those which are optimal among all feasible distributions. As I now demonstrate, all stationary, atomless distributions are efficient.

Theorem 3 *Suppose that μ is an atomless probability measure with support $\bar{A}$. Measure μ results in an efficient allocation if and only if $\mu \in \mathscr{M}$.*

Proof. It is clear that an allocation is efficient if and only if it maximizes aggregate or average fitness. Let $\mu \in \mathscr{M}$. In a market economy, with identical homogeneous fitness functions, all agents consume goods 1 and 2 in the same proportion. Hence if aggregate consumption maximizes aggregate fitness then average fitness is also maximized. With the law of large numbers, the expected production of good 2 is equal to realized production so let σE_2 be the production of good 2 when E_2 is aggregate investment. Optimal E_2 satisfies, $\sigma = f_1/f_2$. This holds at any stationary point since $p = f_1/f_2$ when consumers maximize utility and $p = \sigma$ at all stationary points. Hence aggregate consumption at stationary distributions maximizes aggregate fitness and therefore average fitness.

Conversely, suppose μ is atomless and that average fitness is maximized with population distribution μ. Since μ maximizes average population fitness, $f_1/f_2 = \sigma$. Consumer utility maximization implies that $p = f_1/f_2$. Therefore $p = \sigma$ and $\mu \in \mathscr{M}$. ■

Hence, not only does this evolutionary system converge to a stationary distribution but these stationary population distributions are efficient. Even though all investors are not risk-neutral, efficiency is achieved because in the aggregate, the economy behaves as if the entire population were risk-neutral. In particular, aggregate investment is identical to that which would be chosen by a population of risk-neutrals.[5]

Kihlstrom and Laffont (1979) showed that for a given distribution of preferences, the equilibrium is efficient only if all entrepreneurs are risk-

[5] *As if risk-neutrality* is robust to the notion that the probability of a successful investment can change over time. Take an example where σ alternates between σ_1 and σ_2. Eventually, risk-averse and risk-loving agents alternate between doing well and doing poorly. However, risk-neutral agents always do well.

neutral and hence, for most distributions, their equilibrium is inefficient. In general, for an arbitrary distribution of preferences, the static equilibrium of the current paper is also inefficient. However, the introduction of evolutionary dynamics to the general equilibrium model allows risk preferences to adjust over time, towards an efficient distribution. Obviously, if perfect insurance were available, the static outcome would be efficient and the evolutionary model would play no role. However, perfect insurance markets do not always exist (e.g., moral hazard). In this case, even in the absence of insurance, evolutionary pressures will result in long run efficiency.

6 An example with two types

Assume that there is a risk-averse and a risk-loving type with $\alpha^{RA} < 1 < \alpha^{RL}$ with the proportion of risk-averse and risk-loving types being n^{RA} and n^{RL}. With a distribution consisting of two atoms, it is no longer true that each type chooses the same action so let $\tilde{n}^{RA}$ and $\tilde{n}^{RL}$ be the masses of risk-averse types and risk-loving types that invest.

With two types, the market clearing conditions for goods 1 and 2 are:

$$\begin{aligned} &x_1^*(p^e)[(n^{RA} - \tilde{n}^{RA})p^e + \tilde{n}^{RA}\sigma + (n^{RL} - \tilde{n}^{RL})p^e + \tilde{n}^{RL}\sigma] \\ &\quad = (n^{RA} - \tilde{n}^{RA}) + (n^{RL} - \tilde{n}^{RL}) \\ &x_2^*(p^e)[(n^{RA} - \tilde{n}^{RA})p^e + \tilde{n}^{RA}\sigma + (n^{RL} - \tilde{n}^{RL})p^e + \tilde{n}^{RL}\sigma] \\ &\quad = \sigma(\tilde{n}^{RA} + \tilde{n}^{RL}). \end{aligned} \tag{15}$$

Gross substitutability again implies existence and uniqueness of the competitive equilibrium.

A Nash equilibrium in investments is a triplet $(\tilde{n}^{RA}, \tilde{n}^{RL}, p)$ such that i) $(\tilde{n}^{RA}, \tilde{n}^{RL})$ are such that all agents solve (3), given p, ii) $p = p^e$ where p^e satisfies (15), given $(\tilde{n}^{RA}, \tilde{n}^{RL})$.

Proposition 1 *There exists a unique equilibrium, $(\tilde{n}^{RA}, \tilde{n}^{RL}, p)$, of the investment stage where:*

i) $p \in [\underline{p}, \bar{p}]$ *where* $\underline{p} = \sigma^{1/\alpha^{RA}}$ *and* $\bar{p} = \sigma^{1/\alpha^{RL}}$,

ii) $p \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} \sigma$ *as* $n^{RL} \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} x_2^*(\sigma)$,

iii) *if* $\begin{Bmatrix} p = \underline{p} \\ p \in (\underline{p}, \bar{p}) \\ p = \bar{p} \end{Bmatrix}$

then $\begin{cases} \tilde{n}^{RL} = n^{RL} \text{ and } \tilde{n}^{RA} \text{ is such that } \underline{p} \text{ clears the market.} \\ \tilde{n}^{RA} = 0 \text{ and } \tilde{n}^{RL} = n^{RL}. \\ \tilde{n}^{RA} = 0 \text{ and } \tilde{n}^{RL} \text{ is such that } \bar{p} \text{ clears the market.} \end{cases}$

The two types version of the population dynamics equation is thus:

$$n_{t+1}^{RL} = \frac{\left(n_t^{RL} - \tilde{n}_t^{RL}\right)p_t + \tilde{n}_t^{RL}\sigma}{\left(n_t^{RA} - \tilde{n}_t^{RA}\right)p_t + \tilde{n}_t^{RA}\sigma + \left(n_t^{RL} - \tilde{n}_t^{RL}\right)p_t + \tilde{n}_t^{RL}\sigma} \ . \tag{16}$$

As before, there is a unique stable and stationary distribution that is efficient (i.e., when $n^{RL} = x_2^*(\sigma)$). There are two additional stationary points (0 and 1), however, neither is stable. For example, at $n^{RL} = 0$, $p < \sigma$ so that the monetary payoff to risk-lovers is greater than that of risk-averse agents (i.e., $\sigma > [(n^{RA} - \tilde{n}^{RA})p_t + \tilde{n}^{RA}\sigma]/n^{RA}$ since $\tilde{n}^{RA} < n^{RA}$).

Proposition 2 *When* $n^{RL} = 0(1)$, *the average fitness of risk-averse* (*risk-loving*) *agents decreases if a small contingent of risk-loving* (*risk-averse*) *individuals enters the population.*

Proof. When $n^{RL} = 0$ we know that $p < \sigma$. The average fitness of risk-averse individuals when $p < \sigma$ is $g(p)((n^{RA} - \tilde{n}^{RA})p + \sigma\tilde{n}^{RA})$. When $n^{RL} < \tilde{n}^{RL*}$, $\tilde{n}^{RA} = \tilde{n}^{RL*} - n^{RL}$ so that fitness becomes $g(p)((1 - \tilde{n}^{RL*})p + (\tilde{n}^{RL*} - n^{RL})$. Differentiating with respect to n^{RL} yields:

$$\left[g'(p)\left(\left(1 - \tilde{n}^{RL*}\right)p + \left(\tilde{n}^{RL*} - n^{RL}\right)\sigma + g(p)\left(1 - \tilde{n}^{RL*}\right)\right)\right]\frac{dp}{dn^{RL}} - g(p)\sigma \ .$$

The first term is zero since when $n^{RL} < x_2^*(\sigma)$, price is constant at $\underline{p}$. The second term is always negative and hence for small n^{RL}, the risk-averse types are made worse off by the entry of a small contingent of risk-loving types. The proof for $n^{RL} = 1$ is identical. ■

Although with the introduction of a small contingent of risk-loving mutants, risk-averse agents are made worse off in terms of expected fitness, they are no worse off in terms of expected utility. To see this, note that price is constant at $\underline{p}$, so risk-averse agents are indifferent between investing and not investing and hence they are no worse off, subjectively, by the entry of a small contingent of risk-loving agents.

Finally, a population of risk-averse (risk-loving) types, invaded by risk-lovers (risk-averse), although initially they are made worse off, over time, the population adjusts and:

Proposition 3 *At the stable population distribution, the risk-averse* (*risk-loving*) *agents are better off* (*in both an expected utility and an expected fitness sense*) *than if there were no risk-loving* (*risk-averse*) *individuals.*

Proof. When $n^{RL} < x_2^*(\sigma), p < \sigma$. Since $g(p)$ must be strictly decreasing in p so that $g(p)\sigma > g(\sigma)\sigma$ (i.e., a risk-lover's fitness is greater than her steady state fitness). Recall that average population fitness is maximized when $p = \sigma$. Since at $p < \sigma$, risk-lovers earn fitness greater than their steady state fitness, the risk-averse must be earning below population average fitness which is less than the steady state average fitness. Furthermore, since at the steady state, a risk-averse agent achieves, with certainty, a fitness greater than her expected fitness at $n^{RL} = 0$, her expected utility at the steady state must be greater as well. ■

Thus although risk-averse agents are initially made objectively worse off by the entry of risk-loving mutants, they are better off as the population converges to the steady state.

7 Related literature

The replicator dynamics used in the current paper are closely related to Blume and Easley (1992) where the dynamics are driven by wealth accumulation. To see this, suppose that if type α agents do well, instead of increasing in number, they simply have greater wealth with which to invest. That is, at the end of period t, each type α agent holds either $p_t e_1^*(\alpha, p_t)$ or $p_t e_1^*(\alpha, p_t) + e_2^*(\alpha, p_t)$ units of wealth. On average, type α agents now hold $p_t e_1^*(\alpha, p_t) + \sigma e_2^*(\alpha, p_t)$ of wealth. Suppose $\bar{\mu}_t$ is a measure which describes the distribution of wealth held by each type. The corresponding dynamic is given by:

$$\bar{\mu}_{t+1}(A) = \int_A (p_t e_1^*(\alpha, p_t) + \sigma e_2^*(\alpha, p_t))\, d\bar{\mu}_t(\alpha) \tag{17}$$

for all measurable $A \subset \bar{A}$. Let this wealth dynamic, $\bar{\mu}_{t+1}$, be normalized by average population wealth, $\int_{\bar{A}} (p_t e_1^*(\alpha, p_t) + \sigma e_2^*(\alpha, p_t)) d\bar{\mu}_t(\alpha)$. The resulting dynamic is a probability measure which is precisely the simplified replicator dynamic given by equation (6).

Given a dynamic of this type, Blume and Easley's result is that, for a finite set of agents and provided beliefs are correct and discount factors are uniform, then agents with log utility functions eventually dominate the population. In particular, long run investment behavior is risk-averse. As argued in the introduction, this result depends on the fact that the return on identical portfolios is perfectly correlated. Suppose for example that instead of assets, there are S *types* of investment opportunities and that the return on investment opportunities of the same type are independently and identically distributed. A Blume and Easley model with a continuum of agents would clearly select for agents that are risk-neutral. Thus the degree of risk aversion exhibited in stationary populations depends crucially on the degree of correlation in returns. One would expect that the less the correlation, the closer to risk-neutrality will be the behavior of the population.

Similar conclusions can now be drawn on the Kihlstrom and Laffont (1979) inefficiency result. That is, the long run of a model with i.i.d. risks should find agents behaving, in the aggregate, as if they are risk neutral and thus the economy should exhibit no inefficiency in aggregate investment. However, as the degree of correlation increases, aggregate behavior becomes more and more risk-averse and investment levels become more inefficient. Therefore even when the population distribution is able to adjust in response to market rewards, the resulting allocation is in general not efficient. However, this inefficiency declines with a decline in the degree of correlation in investment outcomes.

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