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Intertemporal Non-separability and Dynamic Oligopoly

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Abstract We construct a framework for modeling dynamic Cournot oligopoly. We consider models where utility maximizing consumers give rise to demand functions that depend on current and prior period prices. Future demand depends on the current price and consumers, and firms must take this into account when making their decisions. Focusing on problems that yield dynamic demand functions that are linear in current and prior period price, we characterize the unique Markov perfect equilibrium in linear strategies. We then demonstrate the applicability of our framework through a series of practical examples.

Keywords Dynamic oligopoly · Durable goods · Habit persistence · Inventories

JEL Classification: C73 · D21 · D43

1 Introduction

Traditionally, dynamic models of oligopoly have assumed that a decision made in the past does not directly affect payoffs in the future. That is, agents' decisions are "time separable" so that single-period utility, budget constraints or profit functions depend only on variables (e.g., consumption of a good) determined in that period. While this greatly simplifies analyses, there are many goods for which this is not realistic. For example, purchases of some goods may be substitutable over time [13], automobiles or household appliances are durable [4,9,14], consumers may be able to hold inventories of non-perishable groceries like canned goods

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[7] or some goods like coffee or cigarettes may be habit forming [1,10]. Indeed, one might argue that "intertemporal non-separability" should be the rule rather than the exception.

Given the pervasiveness of time non-separability, versatile and tractable models are needed both as a guide to empirical work and for theoretical analysis. For instance, time non-separability is important for understanding the persistence of price changes. Standard models of oligopoly have no built-in intertemporal link and without embellishment they admit no persistence. On the other hand, in models with non-time-separability, agents' objective functions are intertemporally linked and persistence arises naturally. For example, if consumption is durable, consumers substitute toward a low current price, implying that next period's demand will be low if this period's price is low. A low current price leads to low future demand, and consequently the future price will be low as well.

There is growing literature that allows for intertemporal non-separabilities. A non-exhaustive list includes models of habit persistence [10,12], durable goods [3,8,9] and inventories [11] . In this paper, we provide a framework in which decisions are intertemporally non-separable and the good can be intertemporally substitutable or complementary. Our framework allows for any degree of market power with an arbitrary number of Cournot competitors and for stochastic shocks, either to firms' costs or to demand. The framework is tractable, and we fully characterize solutions and examine their comparative statics.

To ensure tractability, we consider quadratic utility functions and demonstrate conditions for the uniqueness of Markov perfect equilibria in linear strategies. In equilibrium, prices follow a dynamic stochastic process in which the current price depends both on past prices and on random disturbances. The equilibrium of our model exhibits both persistence and incomplete passthrough. Using several examples, we demonstrate the applicability of our framework to examining intertemporally linked markets.

An important feature of our modeling framework is that we model both the demand and supply side of a dynamic model. When consumers form expectations over the future, a structural change alters not only the decisions of agents directly but also alters implied demand parameters. This may be of significance when evaluating the potential impact of market structure or policy changes.

We begin by laying out the framework in Sect. 2. We then examine a series of examples in Sect. 3. Next, we offer some concluding comments in Sect. 4. Finally, proofs and a supplemental analysis of correlated shocks is given in the "Appendices 1 and 2".

2 The Linear Quadratic Framework

We model product markets with a finite set (N) of infinitely lived firms and a representative consumer (possibly infinitely lived).[1] We assume that the good in question is homogeneous and that firms are Cournot competitors. In each period, given market demand, each firm chooses output to maximize the discounted expected value of profits. Given the market price, consumers make consumption decisions to maximize discounted expected utility.

Assume that the representative consumer has a quadratic utility function that may depend on the state (capital, inventories, etc), current consumption and may in addition depend on prior or next period consumption. We consider models where optimal consumption subject to an intertemporal budget constraint yields a linear, dynamic demand function that depends on the current and prior period price,

[1] It is straightforward to allow for heterogeneity of consumers (we provide two such examples) provided that conditions ensuring nonnegativity of consumption are satisfied.

$$X_t\left(p_t;\, p_{t-1}\right) = \hat{a} - \hat{b} p_t + \hat{d} p_{t-1}. \tag{1}$$

Reduced form parameters, $\hat{a}, \hat{b}$ and $\hat{d}$ are determined in equilibrium and are functions of consumer expectations regarding the future price and "deep" model parameters.[2] We later illustrate with several examples using this basic framework.

The dynamic demand function depends on past prices and the current price. For example, with habit (or addictive) goods or durable goods, purchases of the good in the current period can be either substitutable or complementary with past consumption. Since the price in the previous period affected purchases in the previous period, the dynamic demand function will depend on the previous period's price.

There is another, more subtle way, in which the pricing process affects the dynamic demand function. Namely, through the formation of expectations. In what follows, we will assume that utility functions are quadratic. On the supply side, given linear dynamic demand functions, the solution to firms' profit maximization problem reveals that prices follow a first-order autoregressive process. The autoregression coefficient $\lambda \in (-1, 1)$ depends on reduced form parameters of the dynamic demand function. On the demand side, given that future prices follow an autoregressive process, the solution to the representative consumer's utility maximization problem is a linear dynamic demand function with reduced form parameters that depend on the autoregression coefficient, λ. In other words, the demand and supply sides are tightly interwoven and a consistent solution for one cannot be had without a solution for the other.

The dynamic demand function will depend on other factors through its dependence on λ. For instance, we show that dynamic demand depends on N. This may have important implications for policy analysis. For example, in analyzing how a change in market structure will affect a market, it will not be sufficient to take into account that the demand function depends on the current and previous prices. When the market structure changes, there will be a shift in dynamic demand, not just a movement along it.

In each period, every firm has an identical marginal production cost of c_t that is taken to be independently and identically distributed over time with $c_t = \bar{c} + \varepsilon_t$, $\varepsilon_t \in [-\varepsilon_L, \varepsilon_H]$ and $E_t \varepsilon_{t+1} = 0$.[3] Let $p_t(X_t;\, p_{t-1})$ represent the inverse demand function. Given firm i's output, x_t^i, the output of other firms, $X_t^{-i} = \sum_{j \neq i} x_t^j$, and last period's, price, p_{t-1}, its t period profit function is given by

$$\pi^i\left(x_t^i;\, X_t^{-i}, p_{t-1}, c_t\right) = \left[p_t\left(x_t^i + X_t^{-i};\, p_{t-1}\right) - c_t\right] x_t^i.$$

Since we are considering Cournot equilibria, firm i is taking the other firms' outputs as given so that firm i's choice of x_t^i has a one-to-one correspondence with choice of p_t. Thus, we have firm i choose p_t rather than x_t^i since it summarizes the impact of the current choice on the next period's state. We can therefore rewrite firm i's period t profit function as:

$$\pi^i\left(p_t;\, X_t^{-i}, p_{t-1}, c_t\right) = \left(p_t - c_t\right)\left(X_t\left(p_t;\, p_{t-1}\right) - X_t^{-i}\right).$$

[2] Given quadratic utility functions, there will generally exist an equilibrium with "policy functions" that are linear in the state. We focus on such equilibria.

[3] It is straightforward to allow for i.i.d. idiosyncratic shocks, i.e., $c_{it} = \bar{c} + \varepsilon_{it}$. In this case, rather than taking rival outputs as given, firms must form beliefs over the distribution over rival outputs.

Firms have discount factor β and in each period, t, taking the output of rival firms as given, firm i chooses price to maximize discounted expected profits:

$$\Pi_t^i = E_t \sum_{\tau=t}^{\infty} \beta^{\tau-t} \pi^i \left(p_\tau; X_\tau^{-i}, p_{\tau-1}, c_\tau \right).$$

Since each period's demand depends only on the current and last period's price, p_{t-1} summarizes each firm's relevant information in a Markov perfect equilibrium. The equilibrium price function will depend on p_{t-1}, c_t and $\bar{w}_t$.

The Bellman equation for firm i can be written as:

$$V^i(p_{t-1}, c_t) = \max_{p_t} \left[(p_t - c_t) \left(X_t(p_t, p_{t-1}) - X_t^{-i}(p_{t-1}, c_t) \right) + \beta E_t V^i(p_t, c_{t+1}) \right]$$

where $X_t^{-i}(p_{t-1}, c_t)$ is the equilibrium output of rival firms as a function of last period's price and this period's marginal cost.[4]

For the moment, assume that each firm's equilibrium output is a linear function of last period's price and this period's cost; let $X_t^{-i} = f + gp_{t-1} - hc_t$. This assumption will be subsequently verified when we complete our solution to the model. Using (1), firm i's residual demand can now be written as follows:

$$\begin{aligned} x_t^i &= \hat{a} - \hat{b}p_t + \hat{d}p_{t-1} - X_t^{-i} \\ &= (\hat{a} - f) - \hat{b}p_t + (\hat{d} - g)p_{t-1} + hc_t \end{aligned} \tag{2}$$

Differentiating and applying the envelope condition, the Euler equation is thus

$$\begin{aligned} (p_t - c_t)\frac{\partial x_t^i}{\partial p_t} + x_t^i + \beta E_t V_p^i(p_t, c_{t+1}) = (\hat{a} - f) - 2\hat{b}p_t \\ + (\hat{d} - g)p_{t-1} + (\hat{b} + h)c_t + \beta(\hat{d} - g)E_t(p_{t+1} - c_{t+1}) = 0 \end{aligned} \tag{3}$$

where V_p^i denotes the derivative of V^i with respect to price. Since X_t^{-i} represents the equilibrium outputs of all firms but i, (3) determines the behavior of prices in equilibrium.

This is a stochastic second-order linear difference equation with characteristic equation:

$$\beta\lambda^2 - 2\frac{\hat{b}}{\hat{d} - g}\lambda + 1 = 0. \tag{4}$$

The Euler equation (3) has a solution of the following form:

$$p_t = (1 - \lambda)\bar{p} + \lambda p_{t-1} + \lambda\frac{\hat{b} + h}{\hat{d} - g}(c_t - \bar{c}) \tag{5}$$

where $\bar{p}$ is the long-run expected price and λ is the smaller of the roots found by factorizing the difference equation. Taking expectations, we see that $E_t p_{t+1}$ is a linear function of p_t.

[4] Note that we have specified the problem as one where in equilibrium, rival outputs are functions of the prior period price and an individual firm maximizes discounted expected profits through choice of price. Equivalently, we could invert (1) to get $p_t(x_t^i + X_t^{-i}, p_{t-1})$ and specify an individual firm's problem more traditionally as:

$$V^i(p_{t-1}, c_t) = \max_{x_t^i}[(p_t(x_t^i + X_t^{-i}(p_{t-1}, c_t), p_{t-1}) - c_t)x_t^i + \beta E_t V^i(p_t, c_{t+1})].$$

In equilibrium, for every i, x_t^i is a best response to X_t^{-i} (i.e., x_t^i solves the Bellman equation) and $X_t^{-i} = \sum_{j \neq i} x_t^j$.

The equilibrium value of λ must solve Eq. (4). Provided that $|\hat{b}/(\hat{d}-g)|>1$ and since we are looking for $|\lambda|\in(0,1)$, solving this yields:

$$\lambda=\begin{cases}\dfrac{\hat{b}/(\hat{d}-g)-\sqrt{(\hat{b}/(\hat{d}-g))^2-\beta}}{\beta} & \text{if } \hat{d}>0\\ \dfrac{\hat{b}/(\hat{d}-g)+\sqrt{(\hat{b}/(\hat{d}-g))^2-\beta}}{\beta} & \text{if } \hat{d}<0\end{cases}. \tag{6}$$

Solving for the long-run expected price yields,

$$\bar{p}=\frac{\hat{a}-f}{2\hat{b}-(1+\beta)(\hat{d}-g)}+\frac{(\hat{b}+h)-\beta(\hat{d}-g)}{2\hat{b}-(1+\beta)(\hat{d}-g)}\bar{c}. \tag{7}$$

Thus, λ and $\bar{p}$ are functions of model parameters and $\hat{a},\hat{b},\hat{d}$, f, g and h which are in turn functions of model parameters and λ and $\bar{p}$.

Using (5) and summing (2) over all $j\neq i$ yields equations for f, g and h.

$$f=(N-1)(\hat{a}-f)-(N-1)\hat{b}(1-\lambda)\bar{p}+(N-1)\frac{\lambda\hat{b}(\hat{b}+h)}{\hat{d}-g}\bar{c} \tag{8}$$

$$g=-(N-1)\lambda\hat{b}+(N-1)(\hat{d}-g) \tag{9}$$

$$h=(N-1)\frac{\lambda\hat{b}(\hat{b}+h)}{\hat{d}-g}-(N-1)h \tag{10}$$

We can then solve (9) and (10) for g and h.

$$g=\frac{N-1}{N}(\hat{d}-\lambda\hat{b}) \tag{11}$$

$$h=\frac{(N-1)\lambda\hat{b}^2}{\hat{d}} \tag{12}$$

Substituting (11) and (12) into $\lambda(\hat{b}+h)/(\hat{d}-g)$ and then simplifying yields:

$$\lambda\frac{\hat{b}+h}{\hat{d}-g}=\frac{N\lambda\hat{b}}{\hat{d}}. \tag{13}$$

Thus, (5) becomes:

$$p_{t+1}=(1-\lambda)\bar{p}+\lambda p_t+\frac{N\lambda\hat{b}}{\hat{d}}\varepsilon_{t+1}. \tag{14}$$

That is, prices follow an AR(1) process[5] and price changes persist into the future.

Note from (6) and (11), λ depends on $\hat{b}$ and $\hat{d}$ which typically will in turn depend on λ. As a result, we need to prove the existence of a λ^* which solves these equations. In addition, we would like the solution to be unique and to prove some additional properties of the result.

Theorem 1 (i) *For any $\hat{a},\hat{b}$ and $\hat{d}$ such that $\hat{a}>(\hat{b}-\hat{d})\bar{c}>0$, there exist $\varepsilon_L,\varepsilon_H$ such that the equilibrium as defined by (6), (7), (8), (11), (12) and (14) has $p_t, X_t>0$ for all t, $\bar{p}>\bar{c}$ and $N\lambda\hat{b}/\hat{d}<1$.*

[5] With more complicated shocks and additional information and behavioral assumptions to ensure solvability, prices can be shown to follow more complicated stochastic processes. We examine one such extension in "Appendix 2."

(ii) *Suppose that* $\hat{b}(\lambda)$ *and* $\hat{d}(\lambda)$ *are derived from the solution to the consumer's problem. If* $\hat{b}(\lambda)/\hat{d}(\lambda)$ *is nondecreasing in* λ*, then there exists a unique* λ^* *that solves* $\hat{b}(\lambda)$, $\hat{d}(\lambda)$, (6)*, and* (11).

Proof See "Appendix 1." □

Part (i) provides conditions on $\hat{a}, \hat{b}, \hat{d}$ and $\bar{c}$ such that the processes $\{p_t, X_t\}$ are well behaved. Moreover, price changes have persistent effects ($\lambda \neq 0$) and there is incomplete passthrough ($N\lambda\hat{b}/\hat{d} < 1$); these results arise naturally as a result of intertemporal linkages and imperfect competition. Although the theorem assumes that these "reduced form parameters" are given, it tells us that if the equilibrium values of $\hat{a}$, $\hat{b}$ and $\hat{d}$ satisfy our conditions, then with appropriate bounds on the random error term, prices and output are well behaved.

Part (ii) provides a sufficient condition on the consumer's dynamic demand function that ensures existence and uniqueness of an equilibrium in linear strategies. In particular, if the solution to the consumer's problem yields $\hat{b}(\lambda)/\hat{d}(\lambda)$ that is nondecreasing, then this solution coupled with the solution to the firms' problem (Eqs. (6), (7), (8), (11), (12) and (14)) characterizes the unique Markov perfect equilibrium in linear strategies. Nondecreasing $\hat{b}(\lambda)/\hat{d}(\lambda)$ intuitively implies that through consumer expectations, persistence affects the demand impact of the current price at least as much as it affects the impact of the prior price. This is satisfied for each of the examples we shortly consider.

Given the fully characterized equilibrium, we can also examine behavior of output by substituting the equilibrium price into market demand.

$$X_t = (\hat{a} - \hat{b}(1-\lambda)\bar{p}) + (\hat{d} - \lambda\hat{b})p_{t-1} - \frac{N\lambda\hat{b}^2}{\hat{d}}\varepsilon_t \tag{15}$$

When $\hat{d} > 0$, equilibrium output depends positively on past price but negatively on the cost shock. Following a shock that raises production costs, the good's price rises and output falls. After a shock that lowers costs, the price falls and output rises. These effects are temporary and price and output revert to their means with time. Thus, a response analogous to a business cycle boom or depression is endogenous in the model.

We now demonstrate some of the convergence properties of the equilibrium. In particular,

Corollary 1 *As* N *tends to infinity,* λ^* *tends to zero,* $N\lambda^*\hat{b}(\lambda^*)/\hat{d}(\lambda^*)$ *tends to one and* $\bar{p}$ *tends to* $\bar{c}$.

Proof See "Appendix 1." □

Like the standard Cournot model, as the number of firms becomes large, the long-run equilibrium price approaches expected marginal cost so that prices and output approach perfectly competitive levels. Moreover, persistence becomes negligible.

Demand Shocks

Suppose that there is no cost uncertainty but that demand is subject to observable, additive shocks. Let $X_t = \hat{a} - \hat{b}p_t + \hat{d}p_{t-1} + \xi_t$ where ξ_t is i.i.d. and $E_{t-1}\xi_t = 0$. For ease of notation, let $\bar{c} = 0$. If we now define the equilibrium output of rival firms as $X_t^{-i} = f + gp_{t-1} + k\xi_t$, then firm i's residual demand can be written as

$$x_t^i = (\hat{a} - f) - \hat{b}p_t + (\hat{d} - g)p_{t-1} + (1-k)\xi_t. \tag{16}$$

The Euler equation is

$$(\hat{a} - f) - 2\hat{b}p_t + (\hat{d} - g)p_{t-1} + (1 - k)\xi_t + \beta(\hat{d} - g)E_t p_{t+1} = 0$$

and has solution

$$p_{t+1} - p_t = (1 - \lambda)(\bar{p} - p_t) + \frac{\lambda(1 - k)}{\hat{d} - g}\xi_{t+1}. \tag{17}$$

Using (17) and summing (16) over all $j \neq i$, we get an expression for k (the solutions for f and g remain unchanged):

$$k = -(N - 1)\frac{\lambda\hat{b}(1 - k)}{\hat{d} - g} + (N - 1)(1 - k).$$

Solving this for k yields:

$$k = \frac{1 - \dfrac{\lambda\hat{b}}{\hat{d} - g}}{\dfrac{N}{N - 1} - \dfrac{\lambda\hat{b}}{\hat{d} - g}}$$

We know from the discussion earlier in this section that $\lambda\hat{b}/(\hat{d} - g) < 1$ implying that $k > 0$. By examination it is easy to see that $k < 1$ and that as N tends toward infinity, k tends toward 1. Using the system of equations constructed in the proof of Proposition 1, the shock coefficient of the equilibrium price process, $\lambda(1 - k)/(\hat{d} - g)$, can be rewritten as $q(1 - k)/\hat{b}$. Since q and $\hat{b}$ are bounded, as N tends to infinity, this tends to zero. Since the limiting behavior of λ and $\bar{p}$ is the same as before, prices approach marginal cost.

Again substituting the equilibrium price into aggregate demand,

$$X_t = (\hat{a} - \hat{b}(1 - \lambda)\bar{p}) + (\hat{d} - \lambda\hat{b})p_{t-1} + \left(1 - \frac{\lambda\hat{b}(1 - k)}{\hat{d} - g}\right)\xi_t.$$

Since $\lambda\hat{b}/(\hat{d} - g) < 1$ and $0 < k < 1$, $1 - \lambda\hat{b}(1 - k)/(\hat{d} - g) > 0$ and positive demand shocks result in increases in output. This is in contrast to a model with cost shocks. Following a shock that increases demand, both price and output increases; following a shock that reduces demand, both price and output fall. These effects are again temporary and prices and output revert to their means over time. Further, in the limit, as N tends to infinity, the shock coefficient of the output process, $1 - \lambda\hat{b}(1 - k)/(\hat{d} - g)$, tends to 1.

3 Examples

In this section, we examine a number of examples to illustrate the applicability of our framework to a wide variety of situations that include both intertemporal substitutability and complementarity, allow for both finitely lived and infinitely lived consumers and where consumers can be either homogeneous or heterogeneous.

3.1 Overlapping Generations

Consumers live for two periods, and consumption can be either intertemporally substitutable or complementary.

Each generation has a representative consumer who is born with an endowment of wealth $\bar{w}$ which can be divided between consumption when young, when old and a numéraire that is perfectly substitutable between young and old age. Assume that a consumer born in period t who consumes X_t^y when young, X_t^o when old, and w_t of the numéraire gets utility

$$u\left(X_t^y, X_t^o, w_t\right) = a\left(X_t^y + X_t^o\right) - \frac{b}{2}\left(X_t^{y2} + X_t^{o2}\right) - dX_t^y X_t^o + w_t \tag{18}$$

where $a, b > 0$ and $|d| < b$. The numéraire good can be interpreted as money spent on other goods. The parameter b is an indicator of the elasticity of demand, while the parameter d indicates the degree of substitutability or complementarity between current and future consumption. Large values of $d > 0$ imply greater degrees of substitutability with current and future consumption becoming perfectly substitutable as $d \to b$. Similarly, a large negative value of d indicates a greater degree of complementarity.

First consider an old consumer's problem. Old consumers know the price and their level of consumption when they were young. They also know the current price. Since the numéraire good is perfectly substitutable between periods, consumption of the numéraire can be determined in the second period of life. Hence, in period t, an old consumer, born in period $t-1$, chooses X_{t-1}^o and w_{t-1} to maximize utility, given p_{t-1}, X_{t-1}^y and p_t:

$$\begin{aligned}\max_{X_{t-1}^o, w_{t-1}} u\left(X_{t-1}^y, X_{t-1}^o, w_{t-1}\right) &= a\left(X_{t-1}^y + X_{t-1}^o\right) - \frac{b}{2}\left({X_{t-1}^y}^2 + {X_{t-1}^o}^2\right) \\ &\quad - dX_{t-1}^y X_{t-1}^o + w_{t-1} \\ &\quad \text{subject to: } p_{t-1}X_{t-1}^y + p_t X_{t-1}^o + w_{t-1} \le \bar{w}\end{aligned}$$

Provided that $\bar{w}$ is sufficiently large to ensure positive consumption of the numéraire, this is a straightforward maximization problem which yields old consumer demand as a linear function of consumption from last period and the current price.

$$X_{t-1}^o = \frac{a}{b} - \frac{d}{b}X_{t-1}^y - \frac{1}{b}p_t \tag{19}$$

Consumption of the numéraire is given by the remainder of the endowment which was not spent on consumption (i.e., $w_{t-1} = \bar{w} - p_{t-1}X_{t-1}^y - p_t X_{t-1}^o$).

Now consider the young consumer's problem. The young consumer knows current price p_t and has expectations over the future price p_{t+1} and future consumption. Expectations must be consistent with the firm's profit maximization problem and the old consumer's utility maximization problem. The young consumer's problem is:

$$\begin{aligned}\max_{X_t^y} E_t u\left(X_t^y, X_t^o, w_t\right) &= E_t\left\{a(X_t^y + X_t^o) - \frac{b}{2}(X_t^{y2} + X_t^{o2}) - dX_t^y X_t^o + w_t\right\} \\ &\quad \text{subject to: } p_t X_t^y + E_t\{p_{t+1}X_t^o\} + E_t w_t \le \bar{w}\end{aligned}$$

where expectations over X_t^o are governed by (19). Assume for the moment that $E_t p_{t+1} = (1-\lambda)\bar{p} + \lambda p_t$.[6] Solving the young consumer's problem yields:

$$X_t^y = \frac{a}{b+d} + \frac{d(1-\lambda)\bar{p}}{b^2 - d^2} - \frac{b - d\lambda}{b^2 - d^2}p_t. \tag{20}$$

[6] As shown in the prior section, given linear consumer demand, expected prices will indeed have this form. Linearity of demand will be verified shortly.

Given p_{t-1} and assuming that old consumers behaved optimally when they were young, we can substitute (20) into (19) to get old demand as a function of p_t and p_{t-1}.

$$X_{t-1}^o = \frac{a}{b+d} - \frac{d^2(1-\lambda)\bar{p}}{b(b^2-d^2)} + \frac{d(b-d\lambda)}{b(b^2-d^2)} p_{t-1} - \frac{1}{b} p_t. \tag{21}$$

Finally, summing young and old consumer demand yields aggregate consumer demand.

$$X_t = X_{t-1}^o + X_t^y = \hat{a} - \hat{b} p_t + \hat{d} p_{t-1}$$

where

$$\hat{a} = \frac{2a}{b+d} + \frac{d(1-\lambda)\bar{p}}{b(b+d)},\ \hat{b} = \frac{b(b-d\lambda)+(b^2-d^2)}{b(b^2-d^2)},\ \hat{d} = \frac{d(b-d\lambda)}{b(b^2-d^2)}. \tag{22}$$

If $\bar{p} \geq 0, a, b > 0$ and $|d| < b$, then $\hat{a}, \hat{b} > 0$, $|\hat{d}| < \hat{b}$ and $\text{sign}(\hat{d}) = \text{sign}(d)$.

Since,

$$\frac{\hat{b}}{\hat{d}} = \frac{b}{d} + \frac{b^2-d^2}{d(b-d\lambda)}$$

is strictly increasing in λ, part (ii) of Theorem 1 is satisfied so that there is a unique equilibrium in linear strategies. Moreover, if model parameters are such that part (i) of Theorem 1 is satisfied, then there exist bounds on ε_t that ensure that p_t and X_t are well behaved.

Examining (15), since low p_{t-1} implies high X_{t-1}, it is straightforward to see that for $d > 0\,(d < 0)$, adjacent substitutability (complementarity) holds. For $d < 0$, this is consistent with [12] who find that adjacent complementarity holds when "habit capital" depreciates sufficiently quickly—here, since consumers live for two periods, habit capital lasts for just one period and depreciates completely.

We now demonstrate the derivation of comparative statics. In addition, we examine correlated shocks in "Appendix 2."

Comparative Statics

From Corollary 1, we know the limiting effect of market structure ($N \to \infty$) on persistence (none) and passthrough (full). But for non-limiting market structures or for understanding the effect of the degree of intertemporal substitutability on either persistence or passthrough, we need to be able to conduct comparative static exercises. Despite our complicated analytic solution, we are able derive comparative static results. While we focus on the overlapping generations model, similar methods can be used to derive comparative statics for all of our other examples.

In order to get comparative static results, we construct a simplified system of equations which corresponds one-to-one with our economic model's solution and can be tractably differentiated. Our comparative static results will be given for both $d > 0$ and $d < 0$ but for illustrative purposes, we outline the methodology using the case $d > 0$. Let $q = N\lambda s/(1 + (1-1/N)N\lambda s)$ where $s = \hat{b}/\hat{d}$. Equation (4) can be now rewritten as $\beta\lambda^2 - 2q + 1 = 0$ which has a solution $\lambda = \sqrt{(2q-1)/\beta}$. We now construct the following system of equations[7]:

$$\lambda = m\sqrt{2q-1}, \ m = 1/\sqrt{\beta}$$
$$s = \frac{2r^2 - \lambda r - 1}{r - \lambda}, \ r = \frac{b}{d}$$
$$q = \frac{N\lambda s}{1 + (N-1)s\lambda}$$

This three equation system determines the three endogenous variables λ, s, and q, given the three exogenous variables m, r, and N.

To sign partial derivatives of these endogenous variables with respect to exogenous variables, totally differentiate this system:

$$d\lambda = \theta_1 dq + \theta_2 dm$$
$$ds = \theta_3 d\lambda + \theta_4 dr$$
$$dq = \theta_5 d\lambda + \theta_6 ds + \theta_7 dN.$$

It can be shown that all the θ_i's are positive. Writing this system in matrix form:

$$\begin{bmatrix} 1 & 0 & -\theta_1 \\ -\theta_3 & 1 & 0 \\ -\theta_5 & -\theta_6 & 1 \end{bmatrix} \begin{bmatrix} d\lambda \\ ds \\ dq \end{bmatrix} = \begin{bmatrix} \theta_2 & 0 & 0 \\ 0 & \theta_4 & 0 \\ 0 & 0 & \theta_7 \end{bmatrix} \begin{bmatrix} dm \\ dr \\ dN \end{bmatrix}. \tag{23}$$

Define the degree to which cost shocks are passed on as price changes to be $y = N\lambda s$. Solving the above system of equations, we can show the following comparative statics results:

Proposition 1 *For all admissible parameters,* $d|\lambda|/dN < 0, d|\lambda|/d\beta > 0, d|\lambda|/dr < 0, dy/d\beta > 0, dy/dr < 0$ *and for* $N \geq 3, dy/dN > 0$.

Proof See "Appendix 1." □

For the case where $d > 0$, the persistence of price changes unambiguously falls with N. This is consistent with much of the empirical evidence (e.g., [2,5,6]). To understand the intuition, consider the optimal price path, from the point of view of the firms. Since single-period profits are concave in prices, optimality requires price smoothing. With more than one firm, each firm's output decision exerts an externality on other firms by reducing price smoothing. That is, in addition to the static externality one firm's decision imposes on other firms, there is a dynamic externality. As a result, the degree of price smoothing falls as the number of firms rises.

Similarly, persistence is strictly increasing in β, and decreasing in b/d. For $d > 0$, the counterfactuals of either a higher discount factor or more intertemporal substitutability (i.e., falling b/d) imply that current decisions have a greater impact on the future and again, optimality requires price smoothing; the more important the future, the more important are dynamics and thus price smoothing. Therefore, as the discount factor increases or as consumption becomes more intertemporally substitutable, last period's price will have a greater impact on the current price.

[7] This system is a subset of the system of equations used in Appendix 1 to prove Theorem 1. It is important to bear in mind that this is an artificial system of equations. For example, application of stability conditions to establish properties of the solution (i.e., the correspondence principle in macroeconomics) would be a mistake since stability in the artificial system is meaningless.

Finally, the passthrough effect of a cost shock unambiguously rises with firms' patience and falls as the good becomes less intertemporally substitutable. Furthermore, we show that for $N \geq 3$ the initial effect of a shock is rising in the number of firms. That is, price becomes more flexible as an industry becomes more competitive.

3.2 Durable Goods

Consider an infinitely lived consumer who gets utility from a durable good in each period. Let K_t be the stock of a durable good and w_t be the consumption of a numéraire good at time t. With durability, purchases are substitutable over time—current purchases increase the stock of the durable good so that less needs to be purchased in subsequent periods.

Suppose that in each period, the representative consumer gets utility

$$u(K_t, w_t) = aK_t - \frac{b}{2}{K_t}^2 + w_t$$

where $a, b > 0$. The consumer faces period t budget constraint:

$$p_t\left[K_t - (1-\delta)K_{t-1}\right] + w_t \leq \bar{w}.$$

where δK_{t-1} represents the depreciation of the durable good between periods $t-1$ and t, p_t is the price of new purchases of the durable good, and therefore $X_t = K_t - (1-\delta)K_{t-1}$ gives current purchases of the durable good.

The representative consumer seeks to maximize the following dynamic program:

$$U(K_{t-1}, p_t) = \max_{K_t}\left\{aK_t - \frac{b}{2}{K_t}^2 + \bar{w} - p_t\left[K_t - (1-\delta)K_{t-1}\right] + \beta' E_t U(K_t, p_{t+1})\right\} \tag{24}$$

where β' is the consumer's discount factor. The Euler equation and envelope condition are:

$$a - bK_t - p_t + \beta' E_t U_K\left(K_t, p_{t+1}\right) = 0$$
$$U_K\left(K_{t-1}, p_t\right) = (1-\delta)p_t.$$

Shifting the latter forward a period and substituting, the Euler equation becomes:

$$a - bK_t - p_t + \beta'(1-\delta)E_t p_{t+1} = 0.$$

As before assume that $E_t p_{t+1} = (1-\lambda)\bar{p} + \lambda p_t$ where λ and $\bar{p}$ are, for the moment, taken as given. Substituting this into the Euler equation and solving for K_t yields:

$$K_t = \frac{a}{b} + \frac{\beta'(1-\delta)(1-\lambda)\bar{p}}{b} - \frac{1-\beta'(1-\delta)\lambda}{b}p_t \tag{25}$$

The current demand for new durable goods is thus given by:

$$X_t = K_t - (1-\delta)K_{t-1} = \hat{a} - \hat{b}p_t + \hat{d}p_{t-1} \tag{26}$$

where

$$\hat{a} = \delta\frac{a + \beta'(1-\delta)(1-\lambda)\bar{p}}{b}, \tag{27}$$

$$\hat{b} = \frac{1-\beta'(1-\delta)\lambda}{b} \tag{28}$$

and

$$\hat{d} = (1-\delta)\frac{1-\beta'(1-\delta)\lambda}{b}. \tag{29}$$

It is readily verified that $\hat{a}$, $\hat{b}$, and $\hat{d}$ are all positive. As expected, the durability of the good introduces intertemporal substitution.

Finally, note that since

$$\frac{\hat{b}}{\hat{d}} = \frac{1}{1-\delta} > 0$$

is nondecreasing in λ, Theorem 1 applies, so we know that a solution to (6), (28) and (29) exists and that (14) is the unique equilibrium in linear strategies.

Like [3], we find that for finite N, the long-run price of the durable good is strictly greater than the competitive price so that the long-run stock is less than that under competition. Since it fits within our framework, we could straightforwardly allow for quadratic costs as in [8,9].

3.3 Inventories

Consider a model with an infinitely lived consumer who gets utility from current consumption and from holding inventories. In particular, suppose that the total period t utility from consuming y_t and holding i_t inventories is given by:

$$u(i_t, y_t, w_t) = a_i i_t + a_y y_t - (b_i/2){i_t}^2 - (b_y/2){y_t}^2 + d i_t y_t + w_t.$$

The consumer's budget constraint is:[8]

$$p_t(i_{t+1} - i_t + y_t) + w_t \leq \bar{w}.$$

As before, the numéraire enters the utility function linearly. Assume that $d \geq 0$ under the interpretation that the marginal utility of inventories increases as the rate of consumption increases.

The representative consumer maximizes:

$$U(i_t, p_t) = \max_{i_{t+1}, y_t} \Big\{ a_i i_t + a_y y_t - \frac{b_i}{2}{i_t}^2 - \frac{b_y}{2}{y_t}^2 + d i_t y_t + \bar{w} - p_t\left(i_{t+1} - i_t + y_t\right) + \beta' E_t U\left(i_{t+1}, p_{t+1}\right)\Big\}$$

where β' is the consumer's discount factor and p_t is the price of the good in period t.

The first-order conditions are:

$$a_y - b_y y_t + d i_t - p_t = 0 \tag{30}$$

$$-p_t + \beta'\left(a_i - b_i i_{t+1} + d E_t y_{t+1} + E_t p_{t+1}\right) = 0 \tag{31}$$

These first-order conditions can be taken without regard to expectations over future choices because of the envelope theorem. Solving (30) yields:

$$y_t = \frac{a_y}{b_y} + \frac{d}{b_y} i_t - \frac{1}{b_y} p_t \tag{32}$$

Shifting (32) forward one period and substituting into (31) yields:

$$-\frac{p_t}{\beta'} + a_i - b_i i_{t+1} + d\left(\frac{a_y}{b_y} + \frac{d}{b_y} i_{t+1} - \frac{1}{b_y} E_t p_{t+1}\right) + E_t p_{t+1} = 0 \tag{33}$$

[8] It is straightforward to introduce a cost to holding inventory. Suppose for example that inventory decays by the factor δ so that the following budget constraint becomes: $p_t(i_{t+1} - \delta i_t + y_t) + w_t \leq \bar{w}$.

As before, assume $E_t p_{t+1} = (1-\lambda)\bar{p} + \lambda p_t$. Using this to solve (33) yields:

$$i_{t+1} = \frac{a_i + da_y + (b_y - d)(1-\lambda)\bar{p}}{b_i b_y - d^2} - \frac{b_y/\beta' - \lambda(b_y - d)}{b_i b_y - d^2} p_t \tag{34}$$

From (34) and (32):

$$y_t = B_0 - \frac{1}{b_y} p_t - \frac{d(b_y/\beta' - \lambda(b_y - d))}{b_y(b_i b_y - d^2)} p_{t-1}$$

where B_0 is a constant. Total demand for the good, X_t, is therefore given by:

$$\begin{aligned} X_t &= i_{t+1} - i_t + y_t \\ &= \hat{a} - \hat{b} p_t + \hat{d} p_{t-1} \end{aligned} \tag{35}$$

where

$$\hat{b} = \Omega + \frac{1}{b_y}$$

$$\hat{d} = \Omega \frac{b_y - d}{b_y}$$

$$\Omega = \frac{b_y/\beta' - \lambda(b_y - d)}{b_i b_y - d^2}.$$

If the utility function is concave in y_t and i_{t+1}, Ω is positive (i.e., differentiate (33)). Moreover, assume that $b_y - d > 0$; if $b_y - d < 0$, then an exogenous increase in inventories would implausibly lead to a larger increase in current consumption (see (32)).

Note this implies that $\hat{b}/\hat{d} > 1$ and it can then be shown that:

$$\frac{\hat{b}}{\hat{d}} = \frac{b_y}{b_y - d} + \frac{1}{\Omega(b_y - d)}$$

is strictly increasing in λ so that part (ii) of Theorem 1 is satisfied so that there is a unique equilibrium in linear strategies. When part (i) is satisfied, there exists distributions over ε_t that ensure positive prices and output in every period.

Interestingly, the autoregression in prices is positive here, even though inventories and consumption are complements. The reason is as follows: if the current price is low, the agent increases its inventories. Other things being equal (as long as the increase in y in the next period is smaller than the increase in i) this lowers purchases in the next period. Thus, current and future demands for the good are intertemporal substitutes.

Dudine et al. [11] also consider a model where consumers can hold inventories. They consider a monopoly seller in a market where demand rises in every period. They examine pricing when the monopolist can and cannot pre-commit to future prices and show that in contrast to [4], prices are higher without commitment.

3.4 Durable-Related Consumption

Suppose consumers must choose their consumption of a good, y, that is durable for two periods and consumption of a good, x, that is nondurable but related to good y (either as a substitute or as a complement). For example, x might be fossil fuel purchases and y might be cars or it might be solar panels. For a car purchase, gasoline would be a complement and for solar panels, natural gas would be a substitute.

In any period t, half the consumers are choosing y for both period t and period $t+1$. The other half chose y for periods $t-1$ and t at time $t-1$. The good x is chosen in each period. Since our focus will be on the market for the related good, assume that the periodic cost of y is exogenously given by n. Let the consumer's utility function be given by:

$$u(x_t, y_t, w_t) = a_x x_t - \frac{b_x}{2}{x_t}^2 + a_y y_t - \frac{b_y}{2}{y_t}^2 + d x_t y_t + w_t.$$

The consumer's budget constraint is therefore given by:

$$p_t x_t + n y_t + w_t \leq \bar{w}$$

where $y_t = y_{t-1}$ if the consumer had chosen y in period $t-1$.

The value function for consumers choosing y in period t can be written:

$$U_0(p_t) = \max_{x_t, y_t} \left\{ a_x x_t - \frac{b_x}{2}{x_t}^2 + a_y y_t - \frac{b_y}{2}{y_t}^2 + d x_t y_t - p_t x_t - n y_t + \beta' E_t U_1(p_{t+1}, y_t) \right\}.$$

The value function for consumers who chose y in period $t-1$ is:

$$U_1(p_t, y_{t-1}) = \max_{x_t} \Big\{ a_x x_t - \frac{b_x}{2}{x_t}^2 + a_y y_{t-1} - \frac{b_y}{2}{y_{t-1}}^2 + d x_t y_{t-1} - p_t x_t - n y_{t-1} + \beta' E_t U_0(p_{t+1}) \Big\}.$$

Here, if $d > 0$, the goods are complements, and if $d < 0$, they are substitutes.

From these equations, the demand for x by each group is easily solved and given by:

$$x_t^0 = \frac{a_x}{b_x} + \frac{d}{b_x} y_t - \frac{1}{b_x} p_t, \tag{36}$$

and:

$$x_t^1 = \frac{a_x}{b_x} + \frac{d}{b_x} y_{t-1} - \frac{1}{b_x} p_t. \tag{37}$$

For those choosing y in the current period, the Euler equation is given by:

$$a_y - b_y y_t + d x_t^0 - n + \beta' E_t \{ a_y - b_y y_t + d x_{t+1}^1 - n \} = 0. \tag{38}$$

Substituting the demands for x^0 and x^1 and taking expectations, again assuming that $E_t p_{t+1} = (1-\lambda)\bar{p} + \lambda p_t$, and solving for y_t yield:

$$y_t = \frac{a_y - n + d a_x}{b_y b_x - d^2} - \frac{\beta' d (1-\lambda)\bar{p}}{(1+\beta')(b_y b_x - d^2)} - \frac{(1+\beta'\lambda) d}{(1+\beta')(b_y b_x - d^2)} p_t. \tag{39}$$

Substituting for y in the equations for the demand for x, we get:

$$x_t^0 = A_0 - \left[\frac{1}{b_x} + \frac{d^2(1+\beta'\lambda)}{b_x(1+\beta')(b_y b_x - d^2)} \right] p_t, \tag{40}$$

and:

$$x_t^1 = A_1 - \frac{1}{b_x} p_t - \frac{d^2(1+\beta'\lambda)}{b_x(1+\beta')(b_y b_x - d^2)} p_{t-1}. \tag{41}$$

Total demand in the current period is given by:

$$X_t = x_t^0 + x_t^1 = \hat{a} - \hat{b} p_t + \hat{d} p_{t-1}, \tag{42}$$

where:

$$\hat{d} = -\frac{d^2(1+\beta'\lambda)}{b_x(1+\beta')(b_y b_x - d^2)}, \tag{43}$$

and:

$$\hat{b} = \frac{2}{b_x} - \hat{d}. \tag{44}$$

Note $\hat{d} < 0$ regardless of the sign of d. From these, we have:

$$\frac{\hat{b}}{|\hat{d}|} = 1 + \frac{2(1+\beta')(b_y b_x - d^2)}{d^2(1+\beta'\lambda)}. \tag{45}$$

Note that since $\hat{b}/\hat{d} < 0$ we will have $\lambda < 0$. Note also that $\hat{b}/|\hat{d}| > 1$ and that this will be increasing in $|\lambda|$. Thus, the assumptions of Theorem 1 hold.

Since $\lambda < 0$, x is always an intertemporal complement regardless of whether x and y are complementary or substitutable. If x and y are substitutes, a high price of x in the current period induces consumers choosing y to buy less x and more y. The increase in y in the next period then lowers demand for x in the next period. If x and y are complements, a high price of x in the current period induces consumers choosing y to buy less of both x and y. The lower amount of y in the next period then lowers their demand for x in the next period.

4 Concluding Remarks

In the paper, we constructed a framework for analyzing dynamic oligopoly where current consumption decisions affect future utility. This framework is sufficiently flexible that many types of intertemporal linkages can be modeled, including durable goods, habit persistence and inventories. We show that models that fit into this framework are analytically tractable.

A key feature of our model is that both the demand and supply sides are modeled. This is important because consumers form expectations over the future when making current period choices. If market structure or policy changes, not only do consumer choices change directly but implied demand parameters also change. Without consistent modeling of the consumer decision process, inferences based on a purely supply side model may be misleading.

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Appendix 1: Proof

We only include the proofs for the case when $\hat{d} > 0$. Suitable modifications yield proofs for $\hat{d} < 0$.

Proof of Theorem 1 We begin with the latter part.

First, we define some additional notation. For each N let:

$$Q = \frac{\hat{b}}{\hat{d} - g}\lambda = \frac{N\lambda s}{1 + (1 - 1/N)N\lambda s}$$

where $s = \hat{b}/\hat{d}$. Using this definition, we can rewrite (4) as:

$$\beta\lambda^2 - 2Q + 1 = 0$$

Solving this for λ then yields:

$$\lambda = \sqrt{\frac{2Q-1}{\beta}}.$$

Note that if we find a $\lambda \in (0, 1)$ that satisfies (4), the above implies that $1/2 < Q < (1+\beta)/2$. Now, define the following functions over $1/2 \leq q \leq (1+\beta)/2$:

$$\begin{aligned} \Lambda(q) &= \sqrt{\frac{2q-1}{\beta}} \in [0,1] \\ s(q) &= \left.\frac{\hat{b}}{\hat{d}}\right|_{\lambda=\Lambda(q)} \\ \eta(q) &= \frac{Ns(q)}{\sqrt{\beta}} > N/\sqrt{\beta} > N \\ y(q) &= \eta(q)\sqrt{2q-1} = N\Lambda(q)s(q) \\ \gamma(q) = \frac{y(q)}{1+(1-1/N)y(q)} &= \frac{N\Lambda(q)s(q)}{1+(1-1/N)N\Lambda(q)s(q)} \\ k(q) &= q - \gamma(q). \end{aligned} \tag{46}$$

Suppose we knew an equilibrium value of λ. This would imply a value for Q so that $\lambda = \Lambda(Q)$. Then, by construction, $\hat{b}/\hat{d} = s(Q)$, $N\lambda\hat{b}/\hat{d} = y(Q)$ and $Q = \gamma(Q)$. That is, the Q corresponding to an equilibrium will be a fixed point of $\gamma(\cdot)$. Also, if q^* is a fixed point of $\gamma(\cdot)$, it is easily verified that $\Lambda(q^*)$ satisfies the equilibrium conditions. So equilibria will correspond one-to-one with fixed points of $\gamma(\cdot)$ or, equivalently, to points at which $k(q^*) = 0$. Note that all the above functions are continuous over the range of q and all are nondecreasing in q except $k(\cdot)$. With this notation, we can now turn to the proof.

The strategy of the proof is to show that there is a unique $q^* \in [1/2, (1+\beta)/2]$ such that $k(q^*) = 0$ and $y(q^*) < 1$. Now, we can write:

$$k(q) = \frac{q - \eta(q)\sqrt{2q-1}[1-(1-1/N)q]}{1+(1-1/N)\eta(q)\sqrt{2q-1}}$$

and this will be zero if and only if the numerator is zero. Define, for $1/2 \leq q \leq (1+\beta)/2$ and $\eta > N/\sqrt{\beta}$ (since $\eta(q) > N/\sqrt{\beta}$):

$$\mu(q,\eta) = q - \eta\sqrt{2q-1}[1-(1-1/N)q].$$

The following properties of μ are important: μ is continuous, strictly convex in q, and strictly decreasing in η. By construction, an equilibrium corresponds to a point q^* such that $\mu(q^*, \eta(q^*)) = 0$. Note also that the composite function $\mu(q, \eta(q))$ is continuous.

We next show an equilibrium exists. First:

$$\mu\left(1/2, \eta\left(1/2\right)\right) = 1/2 > 0.$$

Also:

$$\mu\left(\frac{1+\beta}{2},\eta\right)=\frac{1+\beta}{2}-\eta\sqrt{\beta}\left[1-\left(1-\frac{1}{N}\right)\frac{1+\beta}{2}\right]$$
$$<\frac{1+\beta}{2}-N\left[1-\left(1-\frac{1}{N}\right)\frac{1+\beta}{2}\right],\ \forall\eta>\frac{N}{\sqrt{\beta}}$$

so

$$\mu\left(\frac{1+\beta}{2},\eta\left(\frac{1+\beta}{2}\right)\right)<\frac{1+\beta}{2}(1+N-1)-N=N\left(\frac{1+\beta}{2}-1\right)<0$$

since $\beta<1$. By continuity, there exists $q^*\in(1/2,(1+\beta)/2)$ such that $\mu(q^*,\eta(q^*))=0$.

We now show q^* is unique. Take arbitrary $q\in(1/2,q^*)$. Since $q^*<(1+\beta)/2$, take $\alpha\in(0,1)$ such that $q^*=\alpha q+(1-\alpha)(1+\beta)/2$. By convexity:

$$\mu(q^*,\eta(q^*))=0<\alpha\mu(q,\eta(q^*))+(1-\alpha)\mu\left(\frac{1+\beta}{2},\eta(q^*)\right).$$

The second term on the right-hand side is negative (since the above argument showed this was true for arbitrary η), so the first must be positive. Therefore, $\mu(q,\eta(q^*))>0$. But $\eta(q)$ is increasing in q so $\eta(q^*)>\eta(q)$. Since μ is decreasing in η, this implies $\mu(q,\eta(q))>\mu(q,\eta(q^*))>0$, so $\mu(q,\eta(q))>0$ for all $q\in[1/2,q^*)$. That $\mu(q,\eta(q))$ is not zero for $q\in(q^*,(1+\beta)/2]$ is now obvious since otherwise the same argument would imply $\mu(q^*,\eta(q^*))>0$.

Turning to the former part, we first show that $N\lambda\hat{b}/\hat{d}<1$. Note that $N\lambda\hat{b}/\hat{d}=y(q^*)$ and if $q^*=\gamma(q^*)<N/(2N-1)$ it must be that $y(q^*)<1$. So it suffices to show $q^*<N/(2N-1)$. If $N/(2N-1)>(1+\beta)/2$, we are done. Otherwise:

$$\mu\left(\frac{N}{2N-1},\eta\left(\frac{N}{2N-1}\right)\right)=\frac{N}{2N-1}-\eta\left(\frac{N}{2N-1}\right)\frac{1}{\sqrt{N-1}}\left[1-\frac{N-1}{N}\cdot\frac{N}{2N-1}\right]$$
$$=\frac{N}{2N-1}\frac{\sqrt{2N-1}-\eta\left(\frac{N}{2N-1}\right)}{\sqrt{2N-1}}$$
$$<\frac{N}{2N-1}\frac{\sqrt{2N-1}-N}{\sqrt{2N-1}}$$
$$\leq 0$$

where the inequalities hold since $\eta>N$ and $N/(2N-1)\leq 1$, $N\geq\sqrt{2N-1}$.[9]

Finally, solving (8) for f and using (10) yields:

$$f=\frac{N-1}{N}\hat{a}-\frac{N-1}{N}\hat{b}(1-\lambda)\bar{p}+h\bar{c}$$

Substituting this, g and h into $\bar{p}$ and solving yields:

$$\bar{p}-\bar{c}=\frac{\hat{a}-(\hat{b}-\hat{d})\bar{c}}{\hat{b}(1+N-(N-1)\beta\lambda)-\hat{d}(1+\beta)}. \tag{47}$$

It is straightforward to see that as long as $\hat{a}>(\hat{b}-\hat{d})\bar{c}$ it must be the case that $\bar{p}>\bar{c}$.

[9] This can easily be derived from the fact that $(N-1)^2\geq 0$.

Now we show that there exist $\varepsilon_L, \varepsilon_H$ where $p_t, X_t > 0$. Define

$$p_L = \bar{p} - N\frac{\lambda}{1-\lambda}\frac{\hat{b}}{\hat{d}}\varepsilon_L \tag{48}$$

$$p_H = \bar{p} + N\frac{\lambda}{1-\lambda}\frac{\hat{b}}{\hat{d}}\varepsilon_H. \tag{49}$$

It is easy to show that if $p_t \in [p_L, p_H]$, then $p_{t+1} \in (p_L, p_H)$. By induction, if $p_t \in [p_L, p_H]$, then $p_{t+k} \in (p_L, p_H)$ for any $k > 0$. Thus, by bounding the errors we can ensure that prices set by the Euler equation will never be negative and will be lower than some price at which the households would always choose positive consumption in both periods. □

Proof of Corollary 1 Since $1/2 < Q < N/(2N-1)$, Q must tend to $1/2$ as N tends to infinity. Since $\lambda = \Lambda(Q)$ and this function is continuous, λ tends to $\Lambda(1/2) = 0$.

Note Q is a one-to-one, continuous function of $y = N\lambda s$, so the fact that Q converges implies y converges to some limit point y'. But then it must be that $1/2 = y'/(1+y')$. Solving this yields $y' = 1$.

Rearranging (8) and substituting (13), we get

$$f = \frac{N-1}{N}\hat{a} - \frac{N-1}{N}\hat{b}(1-\lambda)\bar{p} + \frac{N-1}{N}\frac{N\lambda\hat{b}}{\hat{d}}\hat{b}\bar{c}.$$

Since we have already shown that $\lambda^* \to 0$ and $N\lambda^*\hat{b}/\hat{d} \to 1$, in the limit as $N \to \infty$, $f \to \hat{a} - \hat{b}(\bar{p} - \bar{c})$ or in other words, $\hat{a} - f \to \hat{b}(\bar{p} - \bar{c})$. It is straightforward to see that $h \to \hat{b}$ and $\hat{d} - g \to 0$ and therefore $\bar{p} \to \bar{c}$. □

Proof of Proposition 1 Consider the case where $d > 0$.[10] The determinant of the matrix on the left-hand side of (23) is $D = 1 - \theta_1(\theta_3\theta_6 + \theta_5)$. Now:

$$\theta_1 = \frac{m}{\sqrt{2q-1}} > \sqrt{2N-1}$$

since $m > 1$ and $q < N/(2N-1)$.

$$\theta_5 = \frac{Ns}{[1+(1-1/N)N\lambda s]^2} > \frac{N}{[(2N-1)/N]^2} = \frac{N^3}{(2N-1)^2}$$

since $s > 1$ and $N\lambda s < 1$. Therefore, $\theta_1\theta_5 > N^3/(2N-1)^{3/2}$. It is possible to show that the last expression equals one when $N = 1$ and is greater than one for $N > 1$. This guarantees that D is negative.

We can now sign the partial derivatives of λ using Cramer's rule:

$$\frac{d\lambda}{dN} = \frac{1}{D}\begin{vmatrix} 0 & 0 & -\theta_1 \\ 0 & 1 & 0 \\ \theta_7 & -\theta_6 & 1 \end{vmatrix} = \frac{\theta_1\theta_7}{D} < 0$$

$$\frac{d\lambda}{dm} = \frac{1}{D}\begin{vmatrix} \theta_2 & 0 & -\theta_1 \\ 0 & 1 & 0 \\ 0 & -\theta_6 & 1 \end{vmatrix} = \frac{\theta_2}{D} < 0$$

$$\frac{d\lambda}{dr} = \frac{1}{D}\begin{vmatrix} 0 & 0 & -\theta_1 \\ \theta_4 & 1 & 0 \\ 0 & -\theta_6 & 1 \end{vmatrix} = \frac{\theta_1\theta_4\theta_6}{D} < 0$$

[10] The case where $d < 0$ is analogous but requires $\lambda = -\sqrt{(2q-1)/\beta}$.

In order to get some idea as to the behavior of the term $N\lambda\hat{b}/\hat{d}$, we will need to get similar comparative static results on s. These are:

$$\frac{ds}{dm} = \frac{1}{D}\begin{vmatrix} 1 & \theta_2 & -\theta_1 \\ -\theta_3 & 0 & 0 \\ -\theta_5 & 0 & 1 \end{vmatrix} = \frac{\theta_2\theta_3}{D} < 0$$

$$\frac{ds}{dr} = \frac{1}{D}\begin{vmatrix} 1 & 0 & -\theta_1 \\ -\theta_3 & \theta_4 & 0 \\ -\theta_5 & 0 & 1 \end{vmatrix} = \frac{\theta_4(1-\theta_1\theta_5)}{D} > 0$$

follows from the fact that $\theta_1\theta_5 > 1$.

$$\frac{ds}{dN} = \frac{1}{D}\begin{vmatrix} 1 & 0 & -\theta_1 \\ -\theta_3 & 0 & 0 \\ -\theta_5 & \theta_7 & 1 \end{vmatrix} = \frac{\theta_1\theta_3\theta_7}{D} < 0$$

Now, differentiating $y = N\lambda s$ with respect to m, e and N yields:

$$\frac{dy}{dm} = Ns\frac{d\lambda}{dm} + N\lambda\frac{ds}{dm} < 0$$

$$\frac{dy}{dr} = Ns\frac{d\lambda}{dr} + N\lambda\frac{ds}{dr} < 0$$

Finally, to sign dy/dN, first note that:

$$\begin{aligned}\frac{dy}{dN} &= \lambda s + N\lambda\frac{ds}{dN} + Ns\frac{d\lambda}{dN} \\ &= \lambda s - N\lambda\frac{\theta_1\theta_3\theta_7}{|D|} - Ns\frac{\theta_1\theta_7}{|D|}\end{aligned}$$

where $|D| = \theta_1(\theta_3\theta_6 + \theta_5) - 1 > 0$. Factoring out $|D|$, we get:

$$\frac{dy}{dN} = \frac{1}{|D|}\left(\theta_1\left[\theta_3\left(\theta_6\lambda s - \theta_7 N\lambda\right) + \theta_5\lambda s - \theta_7 Ns\right] - \lambda s\right)$$

Using the definition of the θs:

$$\frac{dy}{dN} = \frac{1}{|D|}\left(\theta_1\left[\theta_3\left(\frac{N\lambda^2 s}{H^2} - \frac{N\lambda^2 s(1-\lambda s)}{H^2}\right) + \frac{N\lambda s^2}{H^2} - \frac{N\lambda s^2(1-\lambda s)}{H^2}\right] - \lambda s\right)$$

where $H = 1 + (N-1)\lambda s$. So,

$$\begin{aligned}\frac{dy}{dN} &= \frac{1}{|D|}\left(\theta_1\left[\theta_3\frac{N\lambda^3 s^2}{H^2} + \frac{N\lambda^2 s^3}{H^2}\right] - \lambda s\right) \\ &= \frac{1}{|D|}\left(\theta_3\frac{N\lambda^2 s^2}{\beta H^2} + \frac{N\lambda s^3}{\beta H^2} - \lambda s\right)\end{aligned}$$

since $\theta_1 = 1/\beta\lambda$. Factoring out $\lambda s/H^2$ yields:

$$\frac{dy}{dN} = \frac{\lambda s}{|D|H^2}\left(\frac{\theta_3 N\lambda s}{\beta} + \frac{Ns^2}{\beta} - H^2\right)$$

This is positive whenever:

$$\frac{Ns^2}{\beta} > H^2$$

or

$$\frac{s}{\sqrt{\beta}}\sqrt{N} > H.$$

Now:

$$\begin{aligned}
\frac{s}{\sqrt{\beta}}\sqrt{N} - H &> \sqrt{N} - H \text{ (since } \frac{s}{\sqrt{\beta}} > 1) \\
&= \sqrt{N} - 1 - \frac{N-1}{N} N\lambda s \\
&= \sqrt{N} - 1 - \frac{N-1}{N} y \\
&> \sqrt{N} - 1 - \frac{N-1}{N} \text{ (since } y < 1) \\
&= \frac{N\sqrt{N} - 2N + 1}{N}
\end{aligned}$$

The term in the numerator is negative for N between one and two. However, it is positive for $N = 3$ and is strictly increasing in N for $N \geq 3$, so dy/dN is positive for $N \geq 3$. □

Appendix 2: Correlated Shocks

We have considered shocks which are i.i.d. over time. We now consider the case in which there is correlation in the cost shocks. As a simple case, take the overlapping generations example of Sect. 3.1 and suppose there is first-order autoregression in the cost-shock series.

In this case, the same method of solving for a Markov perfect equilibrium with linear pricing strategies does not work. If young consumers form expectations of the next period's price as a linear function of the current price, the firms' problem would be the same as in Sect. 2 and Eq. (14) would still give the equilibrium response of firms to such a strategy by consumers. However, the last term in Eq. (14) involves the term ε_{t+1} which will not have expectation of zero unless the current cost shock is zero. Thus, if consumers were to form expectations of the future price assuming the price sequence is first-order autoregressed, the price sequence firms would choose would be second-order autoregressed and those expectations would be inconsistent.

To solve this problem, we instead assume the young consumers observe only the current price and not the cost shock or the history of prices that occurred before they were born. Even with this simple information set, the expectation of the next period's price will not generally be linear (this depends on the distribution that generates the shocks), so we assume consumers use a least-squares projection to form forecasts of the future price.

Given linear consumer forecasts, the firms' problem remains the same and has a solution of as in (14). Let z_t denote the deviation in the price at time t from the long-run expected price. That is, $z_{t+1} \equiv p_{t+1} - \bar{p} = \lambda(p_t - \bar{p}) + \lambda(N\hat{b}/\hat{d})\varepsilon_{t+1}$. This can be rewritten in the form:

$$z_{t+1} = \lambda z_t + \lambda e_{t+1}$$

where e_t is proportional to ε_t. Since the cost shock follows an AR(1) process:

$$e_{t+1} = \rho e_t + u_{t+1}$$

where ρ is less than one in absolute value and u_t is white noise. Let $P(z_{t+1}|z_t)$ be the projection of z_{t+1} given z_t. Since z_t is known by consumers born on date t, to compute

$P(z_{t+1}|z_t)$, we need to find the projection of e_{t+1} given z_t, $P(e_{t+1}|z_t)$. This takes the form:

$$P(e_{t+1}|z_t) = \phi z_t$$

where

$$\phi = \frac{\mathrm{COV}(z_t, e_{t+1})}{\mathrm{VAR}(z_t)}$$

It can be shown that

$$\phi = \frac{(1-\lambda^2)\rho}{\lambda(1+\rho\lambda)}$$

so that the projection of z_{t+1} on z_t is:

$$P(z_{t+1}|z_t) = \lambda(1+\phi)z_t = \frac{\lambda+\rho}{1+\lambda\rho} z_t \equiv \zeta z_t$$

It is easy to show that ζ as defined above is in the interval $[-1, 1]$ whenever $\rho \in$ [-1, 1] and $\lambda \in$ [-1, 1]. To show existence of an equilibrium when $d > 0$, define the function $G : [-1, 1]^2 \to [-1, 1]^2$ as follows. For any given ρ, let:

$$G_1(\zeta, \lambda) = \frac{\lambda+\rho}{1+\lambda\rho}$$

and let G_2 be the right-hand side of (6) where for the solution to the consumers utility maximization problem, we have replaced the λ's appearing in (22) with ζ's. Here, the subscripts index the two arguments of G. It is straightforward to show that G is continuous and maps the compact, convex set $[-1, 1]^2$ back into itself. Therefore, G has a fixed point. By construction, fixed points of G correspond to Markov perfect equilibria, so an equilibrium exists. The primary difference is that prices now follow a second-order autoregression process.

Using similar informational and behavioral assumptions, other stochastic processes can lead to similar results. For example, if shocks are instead assumed to follow a first-order moving average process, it can be shown that prices will then follow an ARMA(1,1) process.

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