Intertemporal Non-separability and Dynamic Oligopoly

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Abstract

We construct a framework for modeling dynamic Cournot oligopoly. We consider models where utility maximizing consumers give rise to demand functions that depend on current and prior period prices. Future demand depends on the current price and consumers and firms must take this into account when making their decisions. Focusing on quadratic, dynamic indirect utility functions, we characterize the unique Markov perfect equilibrium in linear strategies. We then demonstrate the applicability of our framework through a series of practical examples.

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1 Introduction

Traditionally, dynamic models of oligopoly have assumed that a decision made in the past does not directly affect one-period payoffs in the future. That is, agents' decisions are not "time separable" so that single-period utility, budget constraints or profit functions depend only on variables (e.g., consumption of a good) determined in that period. While this greatly simplifies analyses, there are many goods for which this is not realistic. For example, purchases of some goods may be substitutable over time (Hall, 1988), automobiles or household appliances are durable (Bulow, 1982; Driskill, 2001; Stokey, 1981), consumers may be able to hold inventories of non-perishable groceries like canned goods (Cooper and Haltiwanger, 1990) or some goods like coffee or cigarettes may be habit forming (Becker and Murphy, 1988; Driskill and McCafferty, 2001). Indeed, one might argue that "intertemporal non-separability" should be the rule rather than the exception.

Given the pervasiveness of time non-separability, versatile and tractable models are needed both as a guide to empirical work and for theoretical analysis. For example, time non-separability may be important for understanding the persistence of price changes. Standard models of oligopoly have no built-in intertemporal link and without embellishment they admit no persistence. On the other hand, in models with non-time-separability, agents' objective functions are intertemporally linked and persistence may arise naturally. For example, if consumption is durable, consumers substitute toward a low current price, implying that next period's demand will be low if this period's price is low. A low current price leads to low future demand and consequently the future price will be low as well.

Despite their importance and potential for improving our understanding of phenomena such as the persistence of price changes, there has been little prior work examining analytically tractable models of oligopoly with intertemporal linkages. This is due in part to the difficulty in obtaining analytic solutions. Because of this difficulty, models of dynamic oligopoly often focus on cases where consumers are assumed to be myopic. For example, Karp and Perloff (1993), Kirman and Sobel (1974) and Reynolds (1991) examine models of dynamic oligopoly where demand functions are given and constant over time.[1] Exceptions include Fethke and Jagannathan (1996) and Driskill and McCafferty (2001) where they assume that consumers are forward looking with preferences exhibiting habit persistence and where firms

[1]Kirman and Sobel (1974) examines a dynamic oligopoly where firms can hold inventories. Karp and Perloff (1993) and Reynolds (1991) examine dynamic oligopoly where firms have costs of adjusting output. In neither case are consumers dynamic optimizers.

take into account consumer's dynamic preferences in solving their profit maximization problem.

In this paper, we explicitly model both the demand and supply side of a market in which preferences are non-separable. The good can be temporally substitutable or complementary. On the supply side, we allow for any degree of market power as we have an arbitrary number of Cournot competitors. We allow for stochastic shocks, either to firms' costs or to demand, while still keeping the model tractable enough to characterize its solution in a manner that allows for relatively easy comparative statics.

Using this framework, we demonstrate conditions for the uniqueness of Markov perfect equilibria in linear strategies. In equilibrium, prices follow a dynamic stochastic process in which the current price depends both on past prices and on random disturbances. The equilibrium of our model exhibits both persistence and incomplete passthrough. Using several examples, we demonstrate the applicability of our framework to examining intertemporally linked markets. We then examine an application towards optimal taxation for a revenue maximizing policy maker.

An important feature of our modeling framework is that we model both the demand and supply side of a dynamic model. Without modeling the formation of consumer expectations over the future, structural changes alter the decisions of agents directly but also alter implied demand parameters.[2] This may be of significance when evaluating the potential impact of market structure or policy changes.

We begin by laying out the general framework in Section 2. We then examine a series of examples in Section 3. Finally, we offer some concluding comments in Section 4.

2 The Framework

We model product markets with a finite set (N) of infinitely lived firms and a representative consumer (possibly infinitely lived).[3] We assume that the good in question is homogeneous and that firms are Cournot competitors. In each period, given market demand, each firm chooses output to maximize the discounted expected value of profits. Given the market price, consumers make consumption decisions to maximize discounted expected utility.

[2] This is akin to Lucas' admonition against estimating reduced form model parameters and conducting policy analysis since demand parameters can be expected to change in response to policy changes (Lucas, 1976).

[3] It is straightforward to allow for heterogeneity of consumers (we provide two such examples) provided that conditions ensuring nonnegativity of consumption are satisfied.

Assume the representative consumer has a dynamic indirect utility function that depends on the current and previous period's price and on current wealth; $U(p_t, p_{t-1}, \bar{w}_t)$. The dynamic indirect utility function is derived by solving the current period maximization problem, substituting the equilibrium choices into the sum of discounted direct utility and evaluating expectations over future equilibrium behavior.

Given $U(\cdot)$, we can derive consumer demand using Roy's Identity:

$$X_t(p_t, p_{t-1}, \bar{w}_t) = -\frac{\partial U/\partial p_t}{\partial U/\partial \bar{w}_t}.$$

Assume that X_t is invertible so that $p_t(X_t; p_{t-1}, \bar{w}_t)$ is the market inverse demand function at time t where X_t is total industry output.

In each period, every firm has an identical marginal production cost of c_t that is taken to be independently and identically distributed over time with $c_t = \bar{c} + \varepsilon_t$, $\varepsilon_t \in [-\varepsilon_L, \varepsilon_H]$ and $E_t \varepsilon_{t+1} = 0$.[4] Given firm i's output, x_t^i, the output of other firms, $X_t^{-i} = \sum_{j \neq i} x_t^j$, and last period's, price, p_{t-1}, its t period profit function is given by

$$\pi^i(x_t^i; X_t^{-i}, p_{t-1}, \bar{w}_t, c_t) \equiv [p_t(x_t^i + X_t^{-i}; p_{t-1}, \bar{w}_t) - c_t] x_t^i.$$

Since we are considering Cournot equilibria, firm i is taking the other firms' outputs as given so that firm i's choice of x_t^i has a one-to-one correspondence with choice of p_t. Thus we have firm i choose p_t rather than x_t^i since it summarizes the impact of the current choice on the next period's state. We can therefore rewrite firm i's period t profit function as:

$$\pi^i(p_t; X_t^{-i}, p_{t-1}, \bar{w}_t, c_t) = (p_t - c_t)(X_t(p_t; p_{t-1}, \bar{w}_t) - X_t^{-i}).$$

Firms have discount factor β and in each period, t, taking the output of rival firms as given, firm i chooses price to maximize discounted expected profits:

$$\Pi_t^i = E_t \sum_{\tau=t}^{\infty} \beta^{\tau-t} \pi^i(p_\tau; X_\tau^{-i}, p_{\tau-1}, \bar{w}_\tau, c_\tau).$$

Since each period's demand depends only on the current and last period's price, p_{t-1} summarizes each firm's relevant information in a Markov perfect equilibrium. The equilibrium price function will depend on p_{t-1}, c_t and $\bar{w}_t$.

[4] It is straightforward to allow for i.i.d. idiosyncratic shocks, i.e., $c_{it} = \bar{c} + \varepsilon_{it}$. In this case, rather than taking rival outputs as given, firms must form beliefs over the distribution over rival outputs.

The dynamic indirect utility function depends on past prices and the current price.[5] For example, with habit (or addictive) goods or durable goods, purchases of the good in the current period can be either substitutable or complementary with past consumption. Since the price in the previous period affected purchases in the previous period, the dynamic indirect utility function will depend on the previous period's price.

There is another, more subtle way in which the pricing process affects the dynamic indirect utility function (and therefore the dynamic demand for the good). Namely, through the formation of expectations. In what follows, we will assume that direct utility functions are quadratic. On the supply side, given linear dynamic demand functions, the solution to firms' profit maximization problem reveals that prices follow a first-order autoregressive process. The autoregression coefficient $\lambda \in (-1, 1)$ depends on reduced form parameters of the dynamic demand function. On the demand side, given that future prices follow an autoregressive process, the solution to the representative consumer's utility maximization problem is a linear dynamic demand function with reduced form parameters that depend on the autoregression coefficient, λ. In other words, the demand and supply sides are tightly interwoven and a consistent solution for one cannot be had without a solution for the other.

Moreover, the dynamic demand function will depend on other factors through its dependence on λ. For instance, we show that dynamic demand depends on N, the number of firms in the market. This may have important implications for policy analysis. For example, in analyzing how a change in market structure will affect a market, it will not be sufficient to take into account that the demand function depends on the current and previous prices. When the market structure changes, there will be a shift in dynamic demand, not just a movement along it.

2.1 Quadratic Dynamic Indirect Utility

To attain analytic solutions we assume that in each period, the representative consumer has a quadratic, dynamic indirect utility function that is linear in wealth:

$$U(p_t, p_{t-1}, \bar{w}_t) = C - \hat{a}p_t + \frac{\hat{b}}{2}{p_t}^2 - \hat{d}p_t p_{t-1} + \bar{w}_t.$$

[5]In general, $U(\cdot)$ could depend on prices at higher lags, but we focus on the case where it just depends on p_t and p_{t-1}.

The terms $\hat{a}$, $\hat{b}$, $\hat{d}$ and C are reduced form parameters that are determined in equilibrium and are functions of "deep" model parameters.[6]

Since our indirect utility function is linear in wealth, consumer demand is independent of wealth and is simply:

$$X_t(p_t, p_{t-1}) = -\frac{\partial U}{\partial p_t} = \hat{a} - \hat{b}p_t + \hat{d}p_{t-1}. \quad (1)$$

That is, demand depends on both the current and previous period's price. We later illustrate with several more specific examples, how many types of markets can be captured by this basic framework.

The Bellman equation for firm i can be written as:

$$V^i(p_{t-1}, c_t) = \max_{p_t}[(p_t - c_t)(X_t(p_t, p_{t-1}) - X_t^{-i}(p_{t-1}, c_t)) + \beta E_t V^i(p_t, c_{t+1})]$$

where $X_t^{-i}(p_{t-1}, c_t)$ is the equilibrium output of rival firms as a function of last period's price and this period's marginal cost.

For the moment, assume that each firm's equilibrium output is a linear function of last period's price and this period's cost; let $X_t^{-i} = f + gp_{t-1} - hc_t$. This assumption will be subsequently verified when we complete our solution to the model. Using (1), firm i's residual demand can now be written as follows:

$$\begin{aligned} x_t^i &= \hat{a} - \hat{b}p_t + \hat{d}p_{t-1} - X_t^{-i} \\ &= (\hat{a} - f) - \hat{b}p_t + (\hat{d} - g)p_{t-1} + hc_t \end{aligned} \quad (2)$$

The Euler equation is thus

$$\begin{aligned} (p_t - c_t)\frac{\partial x_t^i}{\partial p_t} + x_t^i + \beta E_t V_p^i(p_t, c_{t+1}) = (\hat{a} - f) - 2\hat{b}p_t + \\ (\hat{d} - g)p_{t-1} + (\hat{b} + h)c_t + \beta(\hat{d} - g)E_t(p_{t+1} - c_{t+1}) = 0 \end{aligned} \quad (3)$$

where V_p^i denotes the derivative of V^i with respect to its first argument. Since X_t^{-i} has been assumed to represent the equilibrium outputs of all firms but i, (3) determines the behavior of prices in equilibrium. This is

[6] Note that since our indirect utility functions include the value of future consumption, our assumption that they are quadratic also depends on the nature of the equilibrium. That is, infinite horizon models typically have multiple equilibria and given quadratic direct utility functions, there will generally exist an equilibrium with "policy functions" that are linear in the state, leading to quadratic value functions. We focus on such equilibria.

a stochastic second-order linear difference equation with a solution of the following form:

$$p_t = (1-\lambda)\bar{p} + \lambda p_{t-1} + \lambda \frac{\hat{b}+h}{\hat{d}-g}(c_t - \bar{c}) \tag{4}$$

where $\bar{p}$ is the long run expected price and λ is the smaller of the roots found by factorizing the difference equation. Taking expectations, we see that $E_t p_{t+1}$ is a linear function of p_t.

The equilibrium value of λ must satisfy:

$$\beta\lambda^2 - 2\frac{\hat{b}}{\hat{d}-g}\lambda + 1 = 0 \tag{5}$$

Provided that $|\hat{b}/(\hat{d}-g)| > 1$ and since we are looking for $|\lambda| \in (0,1)$, solving this yields:

$$\lambda = \begin{cases} \dfrac{\hat{b}/(\hat{d}-g) - \sqrt{(\hat{b}/(\hat{d}-g))^2 - \beta}}{\beta} & \text{if } \hat{d} > 0 \\ \dfrac{\hat{b}/(\hat{d}-g) + \sqrt{(\hat{b}/(\hat{d}-g))^2 - \beta}}{\beta} & \text{if } \hat{d} < 0 \end{cases}. \tag{6}$$

Solving for the long run expected price yields,

$$\bar{p} = \frac{\hat{a}-f}{2\hat{b} - (1+\beta)(\hat{d}-g)} + \frac{(\hat{b}+h) - \beta(\hat{d}-g)}{2\hat{b} - (1+\beta)(\hat{d}-g)}\bar{c}. \tag{7}$$

Thus λ and $\bar{p}$ are functions of model parameters and $\hat{a}, \hat{b}, \hat{d}, f, g$ and h which are in turn functions of model parameters and λ and $\bar{p}$.

Using (4) and summing (2) over all $j \neq i$, yields equations for f, g and h.

$$f = (N-1)(\hat{a}-f) - (N-1)\hat{b}(1-\lambda)\bar{p} + (N-1)\frac{\lambda\hat{b}(\hat{b}+h)}{\hat{d}-g}\bar{c} \tag{8}$$

$$g = -(N-1)\lambda\hat{b} + (N-1)(\hat{d}-g) \tag{9}$$

$$h = (N-1)\frac{\lambda\hat{b}(\hat{b}+h)}{\hat{d}-g} - (N-1)h \tag{10}$$

We can then solve (9) and (10) for g and h.

$$g = \frac{N-1}{N}(\hat{d} - \lambda\hat{b}) \tag{11}$$

$$h = \frac{(N-1)\lambda\hat{b}^2}{\hat{d}} \tag{12}$$

Substituting (11) and (12) into $\lambda(\hat{b}+h)/(\hat{d}-g)$ and then simplifying yields:

$$\lambda\frac{\hat{b}+h}{\hat{d}-g}=\frac{N\lambda\hat{b}}{\hat{d}}. \tag{13}$$

Thus (4) becomes:

$$p_{t+1}=(1-\lambda)\bar{p}+\lambda p_t+\frac{N\lambda\hat{b}}{\hat{d}}\varepsilon_{t+1}. \tag{14}$$

That is, prices follow a first-order autoregressive process[7] and price changes have persistent effects into the future.

Note from (6) and (11), λ depends on $\hat{b}$ and $\hat{d}$ which typically will in turn depend on λ. As a result, we need to prove the existence of a λ^* which solves these equations. In addition, we would like the solution to be unique and to prove some additional properties of the result.

Theorem 1 *i) For any $\hat{a}$, $\hat{b}$ and $\hat{d}$ such that $\hat{a}>(\hat{b}-\hat{d})\bar{c}>0$, there exist $\varepsilon_L,\varepsilon_H$ such that the equilibrium as defined by* (6), (7), (8), (11), (12) *and* (14) *has $p_t, X_t>0$ for all t, $\bar{p}>\bar{c}$ and $N\lambda\hat{b}/\hat{d}<1$.*

ii) Suppose that $\hat{b}(\lambda)$ and $\hat{d}(\lambda)$ are derived from the solution to the consumer's problem. If $\hat{b}(\lambda)/\hat{d}(\lambda)$ is nondecreasing in λ then there exists a unique λ^* that solves $\hat{b}(\lambda)$, $\hat{d}(\lambda)$, (6), *and* (11).

Proof: See Appendix.

The first part provides conditions on $\hat{a}$, $\hat{b}$, $\hat{d}$ and $\bar{c}$ such that the stochastic processes $\{p_t, X_t\}$ are well behaved. Moreover, price changes have persistent effects ($\lambda\neq 0$) and there is incomplete passthrough ($N\lambda\hat{b}/\hat{d}<1$); these results arise naturally as a result of intertemporal linkages and imperfect competition. Although the theorem assumes that these "reduced form parameters" are given, it tells us that if the equilibrium values of $\hat{a}$, $\hat{b}$ and $\hat{d}$ satisfy our conditions, then with appropriate bounds on the random error term, prices and output are well behaved.

The second part provides a condition on the consumer's dynamic indirect utility function that ensures existence and uniqueness of an equilibrium in linear strategies. This condition will be satisfied in each of the examples we shortly consider.

[7] With more complicated shocks and additional information and behavioral assumptions to ensure solvability, prices can be shown to follow more complicated stochastic processes. We examine one such extension in Section 3.1.

We can also examine behavior of output by substituting the equilibrium price into market demand.

$$X_t = (\hat{a} - \hat{b}(1-\lambda)\bar{p}) + (\hat{d} - \lambda\hat{b})p_{t-1} - \frac{N\lambda\hat{b}^2}{\hat{d}}\varepsilon_t$$

When $\hat{d} > 0$, equilibrium output depends positively on past price but negatively on the cost shock. Following a shock that raises production costs, the good's price rises and output falls. After a shock that lowers costs, the price falls and output rises. These effects are temporary and prices and output revert to their means with time. Thus a response analogous to a business cycle boom or depression is endogenous in the model.

We now demonstrate some of the convergence properties of the equilibrium. In particular,

Corollary 1 *As N tends to infinity, λ^* tends to zero, $N\lambda^*\hat{b}(\lambda^*)/\hat{d}(\lambda^*)$ tends to one and $\bar{p}$ tends to $\bar{c}$.*

Proof: See Appendix.

Like the standard Cournot model, as the number of firms becomes large, the long run equilibrium price approaches expected marginal cost so that prices and output approach perfectly competitive levels. Moreover, persistence becomes negligible.

Demand Shocks

Suppose that there is no cost uncertainty but that demand is subject to observable, additive stochastic shocks (e.g., the periodic utility function has a random component). That is, let $X_t = \hat{a} - \hat{b}p_t + \hat{d}p_{t-1} + \xi_t$ where ξ_t is i.i.d. and $E_{t-1}\xi_t = 0$. For ease of notation, let $\bar{c} = 0$. If we now define the equilibrium output of rival firms as $X_t^{-i} = f + gp_{t-1} + k\xi_t$ then firm i's residual demand can be written as

$$x_t^i = (\hat{a} - f) - \hat{b}p_t + (\hat{d} - g)p_{t-1} + (1-k)\xi_t. \tag{15}$$

The Euler equation is

$$(\hat{a} - f) - 2\hat{b}p_t + (\hat{d} - g)p_{t-1} + (1-k)\xi_t + \beta(\hat{d} - g)E_t p_{t+1} = 0$$

and has solution

$$p_{t+1} - p_t = (1-\lambda)(\bar{p} - p_t) + \frac{\lambda(1-k)}{\hat{d} - g}\xi_{t+1}. \tag{16}$$

Using (16) and summing (15) over all $j \neq i$, we get an expression for k (the solutions for f and g remain unchanged):

$$k = -(N-1)\frac{\lambda\hat{b}(1-k)}{\hat{d}-g} + (N-1)(1-k).$$

Solving this for k yields:

$$k = \frac{1 - \dfrac{\lambda\hat{b}}{\hat{d}-g}}{\dfrac{N}{N-1} - \dfrac{\lambda\hat{b}}{\hat{d}-g}}$$

We know from the discussion earlier in this section that $\lambda\hat{b}/(\hat{d}-g) < 1$ implying that $k > 0$. Furthermore, by examination it is easy to see that $k < 1$ and that as N tends towards infinity, k tends towards 1. Using the system of equations constructed in the proof of Proposition 1 (i.e., equation (A1) in the appendix), the shock coefficient of the equilibrium price process, $\lambda(1-k)/(\hat{d}-g)$, can be rewritten as $q(1-k)/\hat{b}$. Since q and $\hat{b}$ are bounded, as N tends to infinity, this tends to zero. Since the limiting behavior of λ and $\bar{p}$ are the same as before, prices approach marginal cost.

Again substituting the equilibrium price into aggregate demand,

$$X_t = (\hat{a} - \hat{b}(1-\lambda)\bar{p}) + (\hat{d} - \lambda\hat{b})p_{t-1} + \left(1 - \frac{\lambda\hat{b}(1-k)}{\hat{d}-g}\right)\xi_t.$$

Since $\lambda\hat{b}/(\hat{d}-g) < 1$ and $0 < k < 1$, $1 - \lambda\hat{b}(1-k)/(\hat{d}-g) > 0$ and positive demand shocks result in increases in output. This is in contrast to a model with cost shocks. Following a shock that increases demand, both price and output increases; following a shock that reduces demand, both price and output fall. These effects are again temporary and prices and output revert to their means over time. Further, in the limit, as N tends to infinity, the shock coefficient of the output process, $1 - \lambda\hat{b}(1-k)/(\hat{d}-g)$, tends to 1.

3 Examples

3.1 Overlapping Generations

We begin with an example where consumption can be either intertemporally substitutable or complementary (e.g., habit formation). Assume that each generation of consumers lives for two discrete time periods. Given a market price, consumers make consumption decisions to maximize discounted expected utility.

Each generation has a single representative consumer who is born with an endowment of wealth $\bar{w}$ which can be divided between consumption when young, consumption when old and a numéraire that is perfectly substitutable between young and old age. Assume that a consumer born in period t who consumes X_t^y when young, X_t^o when old, and w_t of the numéraire gets utility

$$u(X_t^y, X_t^o, w_t) = a(X_t^y + X_t^o) - \frac{b}{2}(X_t^{y2} + X_t^{o2}) - dX_t^y X_t^o + w_t \tag{17}$$

where $a, b > 0$ and $|d| < b$. The numéraire good can be interpreted as money spent on other goods. The parameter b is an indicator of the elasticity of demand while the parameter d indicates the degree of substitutability or complementarity between current and future consumption. Large values of $d > 0$ imply greater degrees of substitutability with current and future consumption becoming perfectly substitutable as $d \to b$. Similarly, large magnitude negative values of d indicate a greater degree of complementarity.

First consider an old consumer's utility maximization problem. Old consumers know the price and their level of consumption when they were young. They also know the current price. Since the numéraire good is perfectly substitutable between periods, consumption of the numéraire can be determined in the second period of life. Hence, in period t, an old consumer, born in period $t-1$ chooses X_{t-1}^o and w_{t-1} to maximize utility, given p_{t-1}, X_{t-1}^y and p_t:

$$\max_{X_{t-1}^o, w_{t-1}} u(X_{t-1}^y, X_{t-1}^o, w_{t-1}) = a(X_{t-1}^y + X_{t-1}^o) - \frac{b}{2}({X_{t-1}^y}^2 + {X_{t-1}^o}^2) - dX_{t-1}^y X_{t-1}^o + w_{t-1}$$

$$\text{subject to: } p_{t-1}X_{t-1}^y + p_t X_{t-1}^o + w_{t-1} \leq \bar{w}$$

Provided that $\bar{w}$ is sufficiently large to ensure positive consumption of the numéraire, this is a straight forward maximization problem which yields old consumer demand as a linear function of consumption from last period and the current price.

$$X_{t-1}^o = \frac{a}{b} - \frac{d}{b}X_{t-1}^y - \frac{1}{b}p_t \tag{18}$$

Consumption of the numéraire is given by the remainder of the endowment which was not spent on consumption (i.e., $w_{t-1} = \bar{w} - p_{t-1}X_{t-1}^y - p_t X_{t-1}^o$).

Now consider the young consumer's problem. The young consumer knows current price p_t, has expectations over the future price p_{t+1} and future consumption. In equilibrium, expectations over the future price must be consistent with the firm's profit maximization problem and expectations over X_t^o and w_t must be consistent with the old consumer's utility maximization problem.

The young consumer's problem is:

$$\max_{X_t^y} \; E_t u(X_t^y, X_t^o, w_t) = E_t\{a(X_t^y + X_t^o) - \frac{b}{2}(X_t^{y2} + X_t^{o2}) - dX_t^y X_t^o + w_t\}$$
$$\text{subject to: } \; p_t X_t^y + E_t\{p_{t+1}X_t^o\} + E_t w_t \leq \bar{w}$$

where expectations over X_t^o are given by (18). Assume for the moment that $E_t p_{t+1} = (1-\lambda)\bar{p} + \lambda p_t$.[8] Solving the young consumer's problem yields:

$$X_t^y = \frac{a}{b+d} + \frac{d(1-\lambda)\bar{p}}{b^2-d^2} - \frac{b-d\lambda}{b^2-d^2}p_t. \tag{19}$$

Given last period's price, p_{t-1} and assuming that old consumers behaved optimally when they were young, we can substitute (19) into (18) to get old demand as a function of p_t and p_{t-1}.

$$X_{t-1}^o = \frac{a}{b+d} - \frac{d^2(1-\lambda)\bar{p}}{b(b^2-d^2)} + \frac{d(b-d\lambda)}{b(b^2-d^2)}p_{t-1} - \frac{1}{b}p_t. \tag{20}$$

Finally, summing young and old consumer demand yields aggregate consumer demand.

$$X_t = X_{t-1}^o + X_t^y = \hat{a} - \hat{b}p_t + \hat{d}p_{t-1}$$

where,

$$\hat{a} = \frac{2a}{b+d} + \frac{d(1-\lambda)\bar{p}}{b(b+d)}, \; \hat{b} = \frac{b(b-d\lambda)+(b^2-d^2)}{b(b^2-d^2)}, \; \hat{d} = \frac{d(b-d\lambda)}{b(b^2-d^2)}. \tag{21}$$

If $\bar{p} \geq 0$, $a, b > 0$ and $|d| < b$ then $\hat{a}, \hat{b} > 0$, $|\hat{d}| < \hat{b}$ and $\text{sign}(\hat{d}) = \text{sign}(d)$.

Since,

$$\frac{\hat{b}}{\hat{d}} = \frac{b}{d} + \frac{b^2-d^2}{d(b-d\lambda)}$$

is strictly increasing in λ, part ii) of Theorem 1 is satisfied so that there is a unique equilibrium in linear strategies. Moreover, if model parameters are such that part i) of Theorem 1 is satisfied then there exist bounds on ε_t that ensure that p_t and X_t are well behaved.

[8] As shown in the prior section, given linear consumer demand, expected prices will indeed have this form. Linearity of demand will be verified shortly.

Comparative Statics

From Corollary 1, we know the limiting effect of market structure on persistence and passthrough (as $N \to \infty$ there is no persistence and cost changes are fully passed through as price changes). But for non-limiting market structures or for understanding the effect of the degree of intertemporal substitutability on either persistence or passthrough, we need to be able to conduct comparative static exercises. Despite our complicated analytic solution, we are able derive comparative static results. While we focus on the overlapping generations model, similar methods can be used to derive comparative statics for our subsequent examples.

Since the system of equations which defines our equilibrium is extremely complicated, in order to get comparative static results, it is necessary to construct a simplified system of equations which corresponds one-to-one with our economic model's solution and can be tractably differentiated. Our comparative static results will be given for both $d > 0$ and $d < 0$ but for illustrative purposes, we outline the methodology using the case $d > 0$. Let $q = N\lambda s/(1 + (1 - 1/N)N\lambda s)$ where $s = \hat{b}/\hat{d}$. Equation (5) can be now rewritten as $\beta\lambda^2 - 2q + 1 = 0$ which has a solution $\lambda = \sqrt{(2q-1)/\beta}$. Using this, we now construct the following system of equations:[9]

$$\lambda = m\sqrt{2q-1}, \; m = 1/\sqrt{\beta}$$
$$s = \frac{2r^2 - \lambda r - 1}{r - \lambda}, \; r = \frac{b}{d}$$
$$q = \frac{N\lambda s}{1 + (N-1)s\lambda}$$

This three equation system determines the three endogenous variables λ, s, and q, given the three exogenous variables m, r, and N.

To sign partial derivatives of these endogenous variables with respect to exogenous variables, totally differentiate this system:

$$d\lambda = \theta_1 dq + \theta_2 dm$$
$$ds = \theta_3 d\lambda + \theta_4 dr$$
$$dq = \theta_5 d\lambda + \theta_6 ds + \theta_7 dN.$$

[9]This system is a subset of the system of equations used in the Appendix to prove Theorem 1. It is important to bear in mind that this is an artificial system of equations. For example, application of stability conditions to establish properties of the solution (i.e., the correspondence principle in macroeconomics) would be a mistake since stability in the artificial system is meaningless.

It can be shown that all the θ_i's are positive. Writing this system in matrix form:

$$\begin{bmatrix} 1 & 0 & -\theta_1 \\ -\theta_3 & 1 & 0 \\ -\theta_5 & -\theta_6 & 1 \end{bmatrix} \begin{bmatrix} d\lambda \\ ds \\ dq \end{bmatrix} = \begin{bmatrix} \theta_2 & 0 & 0 \\ 0 & \theta_4 & 0 \\ 0 & 0 & \theta_7 \end{bmatrix} \begin{bmatrix} dm \\ dr \\ dN \end{bmatrix}. \tag{22}$$

Define the degree to which cost shocks are passed on as price changes to be $y = N\lambda s$. Solving the above system of equations, we can show the following comparative statics results:

Proposition 1 *For all admissible parameters,* $d|\lambda|/dN < 0$*,* $d|\lambda|/d\beta > 0$*,* $d|\lambda|/dr < 0$*,* $dy/d\beta > 0$*,* $dy/dr < 0$ *and* $dy/dN > 0$ *for* $N \geq 3$*.*

Proof: See Appendix.

For the case where $d > 0$, the persistence of price changes unambiguously falls with the number of firms, N. This is consistent with much of the empirical evidence (e.g., Bils and Klenow, 2004; Carlton, 1986; Caucutt *et al.*, 1999). To understand the intuition, consider the optimal price path, from the point of view of the firms. Since single period profits are concave in prices, optimality requires some price smoothing. When there is more than one firm, each firm's output decision exerts an externality on other firms by reducing price smoothing. That is, in addition to the static externality one firm's decision imposes on other firms, there is a dynamic externality. As a result of this externality, the degree of price smoothing falls as the number of firms rises.

Similarly, persistence is strictly increasing in β, the firms' discount factor, and decreasing in r, the degree to which current and future consumption are substitutable or complementary. A higher discount factor or more intertemporal substitutability implies that current decisions have a greater impact on the future and again, optimality requires price smoothing; the more the more important the future, the more important are dynamics and thus price smoothing. Therefore, as the discount factor increases or as consumption becomes more intertemporally substitutable, last period's price will have a greater impact on the current price.

Finally, the passthrough effect of a cost shock unambiguously rises with firms' patience and falls as the good becomes less intertemporally substitutable. Furthermore, we show that for $N \geq 3$ the initial effect of a shock is rising in the number of firms.[10] That is, price become more flexible as an industry becomes more competitive.

[10]Based on extensive simulation exercises, this also appears to be true for $N < 3$.

Correlated Shocks

To this point we have considered shocks which are i.i.d. over time. We can consider the case in which there is correlation in the cost shocks. As a simple case, suppose there is first-order autoregression in the cost-shock series. In this case, the same method of solving for a Markov-perfect equilibrium with linear pricing strategies does not work. If young consumers form expectations of the next period's price as a linear function of the current price, the firms' problem would be the same as in the beginning of Section 3 and equation (14) would still give the equilibrium response of firms to such a strategy by consumers. However, the last term in equation (14) involves the term ε_{t+1} which will not have expectation of zero unless the current cost shock is zero. Thus, if consumers were to form expectations of the future price assuming the price sequence is first-order autoregressed, the price sequence firms would choose would be second-order autoregressed and those expectations would be inconsistent.

To solve this problem, we instead assume the young consumers observe only the current price and not the cost shock or the history of prices that occurred before they were born. Even with this simple information set, the expectation of the next-period's price will not generally be linear (this depends on the distribution that generates the shocks), so we assume consumers use a least-squares projection to form forecasts of the future price.

Given that consumers use linear forecasts, the firms' problem remains the same and has a solution of the form given in equation (14). Let z_t denote the deviation in the price at time t from the long-run expected price. That is, $z_{t+1} \equiv p_{t+1} - \bar{p} = \lambda(p_t - \bar{p}) + \lambda(N\hat{b}/\hat{d})\varepsilon_{t+1}$. This can be rewritten in the form:

$$z_{t+1} = \lambda z_t + \lambda e_{t+1}$$

where e_t is proportional to the cost shock in period t. By assumption the cost shock follows a first-order autoregression process, so that:

$$e_{t+1} = \rho e_t + u_{t+1}$$

where the scalar ρ is less than one in absolute value and u_t is white noise. Let $P(z_{t+1}|z_t)$ be the projection of z_{t+1} given z_t. Since z_t is known by consumers born on date t, in order to compute $P(z_{t+1}|z_t)$, we need to find the projection of e_{t+1} given z_t, $P(e_{t+1}|z_t)$. This takes the form:

$$P(e_{t+1}|z_t) = \phi z_t$$

where

$$\phi = \frac{COV(z_t, e_{t+1})}{VAR(z_t)}$$

It can be shown that

$$\phi = \frac{(1-\lambda^2)\rho}{\lambda(1+\rho\lambda)}$$

so that the projection of z_{t+1} on z_t is:

$$P(z_{t+1}|z_t) = \lambda(1+\phi)z_t = \frac{\lambda+\rho}{1+\lambda\rho}z_t \equiv \zeta z_t$$

It is easy to show that ζ as defined above is in the interval $[-1,1]$ whenever $\rho \in [-1,1]$ and $\lambda \in [-1,1]$. To show existence of an equilibrium when $d > 0$, define the function $G : [-1,1]^2 \to [-1,1]^2$ as follows. For any given ρ, let:

$$G_1(\zeta, \lambda) = \frac{\lambda+\rho}{1+\lambda\rho}$$

and let G_2 be the right-hand side of (6) where for the solution to the consumers utility maximization problem, we have replaced the λ's appearing in (21) with ζ's. Here the subscripts index the two arguments of G. It is straightforward to show that G is continuous and maps the compact, convex set $[-1,1]^2$ back into itself. Therefore G has a fixed point. By construction, fixed points of G correspond to Markov perfect equilibria, so an equilibrium exists. The primary difference is that prices now follow a second-order autoregression process.

Using similar informational and behavioral assumptions, other stochastic processes can lead to similar results. For example, if shocks are instead assumed to follow a first-order moving average process, it can be shown that prices will then follow an ARMA(1,1) process. Of course more complicated stochastic shocks lead to more complicated price processes.

3.2 Durable Goods

Consider an infinitely lived consumer that consumes a durable good in each period. Let K_t be the stock of a durable good and w_t be the consumption of a numéraire good at time t. Suppose that in each period, the representative consumer gets utility

$$u(K_t) = aK_t - \frac{b}{2}{K_t}^2 + w_t$$

where $a, b > 0$. The consumer faces period t budget constraint:

$$p_t[K_t - (1-\delta)K_{t-1}] + w_t \leq \bar{w}.$$

where δ is the factor that determines how quickly the durable good depreciates, p_t is the price of new purchases of the durable good, and therefore $K_t - (1-\delta)K_{t-1}$ gives current purchases of the durable good, X_t.

The representative consumer seeks to maximize the following dynamic program:

$$U(K_{t-1}, p_t) = \max_{K_t} \left\{ aK_t - \frac{b}{2}{K_t}^2 + \bar{w} - p_t[K_t - (1-\delta)K_{t-1}] + \beta' E_t U(K_t, p_{t+1}) \right\} \tag{23}$$

where β' is the consumer's discount factor. The Euler equation and envelope condition for this problem are:

$$a - bK_t - p_t + \beta' E_t U_K(K_t, p_{t+1}) = 0$$

$$U_K(K_{t-1}, p_t) = (1-\delta)p_t.$$

Shifting the envelope condition forward a period and substituting, the Euler equation becomes:

$$a - bK_t - p_t + \beta'(1-\delta)E_t p_{t+1} = 0.$$

As before assume that $E_t p_{t+1} = (1-\lambda)\bar{p} + \lambda p_t$ where λ and $\bar{p}$ are, for the moment, taken as given. Substituting this into the Euler equation and solving for K_t yields:

$$K_t = \frac{a}{b} + \frac{\beta'(1-\delta)(1-\lambda)\bar{p}}{b} - \frac{1-\beta'(1-\delta)\lambda}{b} p_t \tag{24}$$

The current demand for new durable goods is thus given by:

$$X_t = K_t - (1-\delta)K_{t-1} = \hat{a} - \hat{b}p_t + \hat{d}p_{t-1} \tag{25}$$

where

$$\hat{a} = \delta \frac{a + \beta'(1-\delta)(1-\lambda)\bar{p}}{b}, \tag{26}$$

$$\hat{b} = \frac{1-\beta'(1-\delta)\lambda}{b} \tag{27}$$

and

$$\hat{d} = (1-\delta)\frac{1-\beta'(1-\delta)\lambda}{b}. \tag{28}$$

It is readily verified that $\hat{a}$, $\hat{b}$, and $\hat{d}$ are all positive. As expected, the durability of the good introduces inter-temporal substitution. It is also easily verified that computing and differentiating the dynamic indirect utility function also yields the above demand function.

Finally, note that since

$$\frac{\hat{b}}{\hat{d}} = \frac{1}{1-\delta} > 0$$

is nondecreasing in λ, Theorem 1 applies so we know that a solution to (6), (27) and (28) exists and that (14) is the unique equilibrium in linear strategies.

3.3 Inventories

Consider a model with an infinitely lived consumer who gets utility from current consumption and from holding inventories. In particular, suppose that the total period t utility from consuming y_t and holding i_t inventories is given by:

$$u(i_t, y_t) = a_i i_t + a_y y_t - (b_i/2) i_t^2 - (b_y/2) y_t^2 + d i_t y_t + w_t.$$

The consumer's budget constraint is:[11]

$$p_t(i_{t+1} - i_t + y_t) + w_t \leq \bar{w}.$$

As before, the numéraire enters the utility function linearly. We assume that $d > 0$ under the following interpretation: one would expect the marginal utility of inventories to increase as the rate of consumption increases. Therefore, the representative consumer maximizes:

$$U(i_t, p_t) = \max_{i_{t+1}, y_t} \left\{ a_i i_t + a_y y_t - \frac{b_i}{2} i_t^2 - \frac{b_y}{2} y_t^2 + d i_t y_t + \bar{w} - p_t(i_{t+1} - i_t + y_t) + \beta' E_t U(i_{t+1}, p_{t+1}) \right\}$$

where β' is the consumer's discount factor and p_t is the price of the good in period t.

In each period t the agent chooses y_t and i_{t+1}. The first-order conditions are:

$$a_y - b_y y_t + d i_t - p_t = 0 \tag{29}$$

$$-p_t + \beta'(a_i - b_i i_{t+1} + d E_t y_{t+1} + E_t p_{t+1}) = 0 \tag{30}$$

[11] It is straightforward to introduce a cost to holding inventory. Suppose for example that inventory decays by the factor δ so that the following budget constraint becomes: $p_t(i_{t+1} - \delta i_t + y_t) + w_t \leq \bar{w}$.

These first-order conditions can be taken without regard to expectations over future choices because of the envelope theorem. Now, solving (29) yields:

$$y_t = \frac{a_y}{b_y} + \frac{d}{b_y} i_t - \frac{1}{b_y} p_t \tag{31}$$

Shifting (31) forward one period and substituting into (30) yields:

$$-\frac{p_t}{\beta'} + a_i - b_i i_{t+1} + d\left(\frac{a_y}{b_y} + \frac{d}{b_y} i_{t+1} - \frac{1}{b_y} E_t p_{t+1}\right) + E_t p_{t+1} = 0 \tag{32}$$

As before, assume $E_t p_{t+1} = (1-\lambda)\bar{p} + \lambda p_t$. Using this to solve (32) yields:

$$i_{t+1} = \frac{a_i + d a_y + (b_y - d)(1-\lambda)\bar{p}}{b_i b_y - d^2} - \frac{b_y/\beta' - \lambda(b_y - d)}{b_i b_y - d^2} p_t \tag{33}$$

From (33) and (31):

$$y_t = B_0 - \frac{1}{b_y} p_t - \frac{d(b_y/\beta' - \lambda(b_y - d))}{b_y(b_i b_y - d^2)} p_{t-1}$$

where B_0 is a constant. Total demand for the good, X_t, is therefore given by:

$$\begin{aligned} X_t &= i_{t+1} - i_t + y_t \\ &= \hat{a} - \hat{b} p_t + \hat{d} p_{t-1} \end{aligned} \tag{34}$$

where

$$\begin{aligned} \hat{b} &= \Omega + \frac{1}{b_y} \\ \hat{d} &= \Omega \frac{b_y - d}{b_y} \\ \Omega &= \frac{b_y/\beta' - \lambda(b_y - d)}{b_i b_y - d^2} \end{aligned}$$

On the assumption that the utility function is concave in y_t and i_{t+1}, Ω is positive (i.e., differentiate (32)). Moreover, assume that $b_y - d > 0$; if $b_y - d < 0$ then an exogenous increase in inventories would implausibly lead to a larger increase in current consumption (see (31)).

Note this implies that $\hat{b}/\hat{d} > 1$ and it can then be shown that:

$$\frac{\hat{b}}{\hat{d}} = \frac{b_y}{b_y - d} + \frac{1}{\Omega(b_y - d)}$$

is strictly increasing in λ so that part ii) of Theorem 1 is satisfied so that there is a unique linear equilibrium. When part i) is satisfied, there exists distributions over ε_t that ensure positive prices and output in every period.

Note that the autoregression in prices is positive here, even though inventories and consumption are complements. The reason is as follows: if the current price is low, the agent increases its inventories. Other things being equal (as long as the increase in y in the next period is smaller than the increase in i) this lowers purchases in the next period. Thus, current and future demand for the good are intertemporal substitutes.

3.4 Durable-Related Consumption

Suppose consumers must choose their consumption of a good, y, that is durable for two periods. For example, the purchase of a car or the installation of solar water heating panels. Let x the consumption of a nondurable related good (either a substitute or complement for y). For example, x might be fossil fuel purchases. In the former case, gasoline would be a complement and in the latter case, natural gas would be a substitute.

In any period t, half the consumers are choosing y for both period t and period $t+1$. The other half chose y for periods $t-1$ and t at time $t-1$. The good x is chosen in each period. Since our focus will be on the market for the related good, assume that the price of y is exogenously given by n. Let the consumer's utility function be given by:

$$a_x x_t - \frac{b_x}{2} x_t^2 + a_y y_t - \frac{b_y}{2} y_t^2 + d x_t y_t + w_t.$$

The consumer's budget constraint is therefore given by:

$$p_t x_t + n y_t + w_t \leq \bar{w}$$

where $y_t = y_{t-1}$ if the consumer had chosen y in period $t-1$.

The value function for consumers choosing y in period t can be written:

$$U_0(p_t) = \max_{x_t, y_t} \{a_x x_t - \frac{b_x}{2} x_t^2 + a_y y_t - \frac{b_y}{2} y_t^2 + d x_t y_t - p_t x_t - n y_t + \beta' E_t U_1(p_{t+1}, y_t)\}.$$

The value function for consumers who chose y in period $t-1$ is:

$$U_1(p_t, y_{t-1}) = \max_{x_t} \{a_x x_t - \frac{b_x}{2} x_t^2 + a_y y_{t-1} - \frac{b_y}{2} y_{t-1}^2 + d x_t y_{t-1} - p_t x_t - n y_{t-1} + \beta' E_t U_0(p_{t+1})\}.$$

Here, if $d > 0$ the goods are complements, if $d < 0$ they are substitutes.

From these equations the demand for x by each group is easily solved and given by:

$$x_t^0 = \frac{a_x}{b_x} + \frac{d}{b_x} y_t - \frac{1}{b_x} p_t, \tag{35}$$

and:

$$x_t^1 = \frac{a_x}{b_x} + \frac{d}{b_x} y_{t-1} - \frac{1}{b_x} p_t. \tag{36}$$

For those choosing y in the current period, the Euler equation is given by:

$$a_y - b_y y_t + d x_t^0 - n + \beta' E_t\{a_y - b_y y_t + d x_{t+1}^1 - n\} = 0. \tag{37}$$

Substituting the demands for x^0 and x^1 and taking expectations, again assuming that $E_t p_{t+1} = (1-\lambda)\bar{p} + \lambda p_t$, and solving for y_t yields:

$$y_t = \frac{a_y - n + d a_x}{b_y b_x - d^2} - \frac{\beta' d (1-\lambda)\bar{p}}{(1+\beta')(b_y b_x - d^2)} - \frac{(1+\beta'\lambda)d}{(1+\beta')(b_y b_x - d^2)} p_t. \tag{38}$$

Substituting for y in the equations for the demand for x we get:

$$x_t^0 = A_0 - \left[\frac{1}{b_x} + \frac{d^2(1+\beta'\lambda)}{b_x(1+\beta')(b_y b_x - d^2)}\right] p_t, \tag{39}$$

and:

$$x_t^1 = A_1 - \frac{1}{b_x} p_t - \frac{d^2(1+\beta'\lambda)}{b_x(1+\beta')(b_y b_x - d^2)} p_{t-1}. \tag{40}$$

Total demand in the current period is given by:

$$X_t = x_t^0 + x_t^1 = \hat{a} - \hat{b} p_t + \hat{d} p_{t-1}, \tag{41}$$

where:

$$\hat{d} = -\frac{d^2(1+\beta'\lambda)}{b_x(1+\beta')(b_y b_x - d^2)}, \tag{42}$$

and:

$$\hat{b} = \frac{2}{b_x} - \hat{d}. \tag{43}$$

Note $\hat{d} < 0$ regardless of the sign of d. From these, we have:

$$\frac{\hat{b}}{|\hat{d}|} = 1 + \frac{2(1+\beta')(b_y b_x - d^2)}{d^2(1+\beta'\lambda)}. \tag{44}$$

Note that, since $\hat{b}/\hat{d} < 0$ we will have $\lambda < 0$. Note also that $\hat{b}/|\hat{d}| > 1$ and that this will be increasing in $|\lambda|$. Thus, the assumptions of Theorem 1 hold.

It is interesting to note that x is always an intertemporal complement regardless of whether x and y are complementary or substitutable. If x and y are substitutes, a high price of x in the current period induces consumers choosing y to buy less x and more y. The increase in y in the next period then lowers demand for x in the next period. If x and y are complements, a high price of x in the current period induces consumers choosing y to buy less of both x and y. The lower amount of y in the next period then lowers their demand for x in the next period.

3.5 An Application to Optimal Taxation

Suppose there is a revenue maximizing policy maker that chooses a specific tax to be imposed on a monopolist (i.e., $N = 1$). We assume that the policy maker chooses the tax either without or prior to observing the realization of c_t. In each period, the structure of the game is: i) the policy maker chooses a specific tax, τ_t, ii) the monopolist chooses price and iii) consumers consume. Since the policy maker does not observe c_t, in a Markov equilibrium the tax will be a function of only the prior period's price.

The Monopolist

We begin with the monopolist's problem, given τ_t. The monopolist's Bellman equation is:

$$V(p_{t-1}, c_t, \tau_t) = \max_{p_t}\{(p_t - c_t - \tau_t)X_t(p_t, p_{t-1}) + \beta E_t V(p_t, c_{t+1}, \tau_{t+1})\}.$$

In solving this problem, the monopolist must form a hypothesis about its expectations over the government's tax policy. In particular, suppose that it is linear in the prior period price:

$$\tau_t = l + mp_{t-1}$$

where l and m are constants. This supposition will be confirmed once we solve the government's optimal tax problem.

In order to solve the monopolist's problem, we must substitute the policy maker's expected tax. However, this results in the problem that our solution to the monopolist's profit maximization problem will no longer depend on the current tax. For this reason, we introduce a term representing the policy maker's one period deviation, v_t, from the equilibrium tax. That is,

$$\tau_t = l + mp_{t-1} + v_t.$$

The monopolist's optimal response to v_t is equivalent to its response to τ_t (i.e., $\partial p_t/\partial v_t = \partial p_t/\partial \tau_t$) which will be all we need to solve the government's optimal tax problem. In equilibrium there are no deviations so that $v_t = 0$ for all t.

The monopolist's Euler equation is:

$$(p_t - c_t - \tau_t)\frac{\partial X_t^i}{\partial p_t} + X_t^i + \beta E_t\left[V_p^i(p_t, c_{t+1}, \tau_{t+1}) + V_\tau^i(p_t, c_{t+1}, \tau_{t+1})\frac{\partial \tau_{t+1}}{\partial p_t}\right] =$$
$$\hat{a}(1-\beta m) + (\hat{b}-\beta\hat{d})l - \beta\hat{d}\bar{c} + \hat{b}c_t + \hat{b}v_t + \beta(m\hat{b}+\hat{d})p_{t+1} - 2(\hat{b}+\beta m\hat{d})p_t + (m\hat{b}+\hat{d})p_{t-1} = 0.$$

Using methods similar to those in prior sections, this has solution:

$$p_t = (1-\lambda)\bar{p} + \lambda p_{t-1} + \frac{\lambda\hat{b}}{m\hat{b}+\hat{d}}(c_t - \bar{c}) + \frac{\lambda\hat{b}}{m\hat{b}+\hat{d}}v_t \tag{45}$$

where

$$\bar{p} = \frac{(1-\beta m)\hat{a} + (\hat{b}-\beta\hat{d})(\bar{c}+l)}{2(\hat{b}+\beta m\hat{d}) - (1+\beta)(m\hat{b}+\hat{d})}$$

and

$$\lambda = \begin{cases} \dfrac{\frac{\hat{b}+\beta m\hat{d}}{m\hat{b}+\hat{d}} - \sqrt{\left(\frac{\hat{b}+\beta m\hat{d}}{m\hat{b}+\hat{d}}\right)^2 - \beta}}{\beta} & \text{if } \frac{\hat{b}+\beta m\hat{d}}{m\hat{b}+\hat{d}} > 0 \\ \dfrac{\frac{\hat{b}+\beta m\hat{d}}{m\hat{b}+\hat{d}} + \sqrt{\left(\frac{\hat{b}+\beta m\hat{d}}{m\hat{b}+\hat{d}}\right)^2 - \beta}}{\beta} & \text{if } \frac{\hat{b}+\beta m\hat{d}}{m\hat{b}+\hat{d}} < 0 \end{cases}.$$

In equilibrium, $v_t = 0$ so that,

$$p_t = (1-\lambda)\bar{p} + \lambda p_{t-1} + \frac{\lambda\hat{b}}{m\hat{b}+\hat{d}}(c_t - \bar{c}).$$

Notice that without government intervention (i.e., $m = 0$), this is identical to our earlier solution (4).

The Government

The government's problem is to maximize total discounted expected revenues. The Bellman equation for the government is therefore:

$$R(p_{t-1}) = \max_{\tau_t} E_t[\tau_t X_t(p_t, p_{t-1}) + \beta'' R(p_t)]$$

where β'' is the government's discount factor. The government's Euler equation is therefore:

$$E_t\left[\tau_t \frac{\partial X_t}{\partial p_t}\frac{\partial p_t}{\partial \tau_t} + X_t + \beta'' R'(p_t)\frac{\partial p_t}{\partial \tau_t}\right] = 0.$$

Substituting for X_t and p_t and then solving for τ_t yields:

$$\tau_t = \frac{\hat{a} + \beta'' l(\hat{d} - \hat{b}\lambda)o}{\hat{b}o} - \frac{[(1 + \beta'' m\lambda o)\hat{b} - \beta'' mo\hat{d}](1 - \lambda)\bar{p}}{\hat{b}o} + \frac{(1 + \beta'' mo\lambda)(\hat{d} - \hat{b}\lambda)}{\hat{b}o} p_{t-1} \tag{46}$$

Where $o = \partial p_t / \partial \tau_t = \lambda \hat{b} / (m\hat{b} + \hat{d})$. In other words, the solution to the government's problem is linear in p_{t-1}, confirming our earlier assumption.

Solving for the coefficient m yields

$$m = \frac{\hat{d}(\hat{d} - \hat{b}\lambda)}{\hat{b}[(2 + \beta''\lambda^2)\hat{b}\lambda - (1 + \beta''\lambda^2)\hat{d}]}. \tag{47}$$

Now consider the model of durable goods so that $s = \hat{b}/\hat{d}$ can be treated as a constant. In this case,

Proposition 2 *For the durable goods model of Section 3.2, there exists an equilibrium monopoly price, p_t, and optimal tax, τ_t, that follow* (45) *and* (46).

Proof: See appendix.

That is, there exists an equilibrium of the optimal tax model where the monopolist's price policy is linear in the prior period price and current cost, and the government's tax policy is linear in the prior period price. Moreover, it can be shown that if the depreciation rate δ is not too large then the optimal tax is increasing in last period's price.[12] Finally, the problem here is somewhat more complicated than our prior examples and thus we are unable to prove uniqueness.

4 Concluding Remarks

In the paper, we constructed a model of dynamic oligopoly where current consumption decisions affect future utility. This framework is sufficiently flexible that many types of intertemporal linkages can be modeled, including durable goods, habit persistence and inventories. In addition, our model can be used to look at important policy questions such as optimal taxation. Moreover, we show that models that fit into this framework are analytically tractable.

[12] While we have not been able to prove that $m > 0$ generally, numeric computations suggest that this is the case.

A key feature of our model is that both the demand and supply sides are modeled. This is important because consumers form expectations over the future when making current period choices. If market structure or policy changes, not only do consumer choices change directly but implied demand parameters also change. Without consistent modeling of the consumer decision process, inferences based on a purely supply side model may be misleading.

Appendix

A Proofs

We only include the proofs for the case when $\hat{d} > 0$. Suitable modifications yield proofs for $\hat{d} < 0$.

Proof of Theorem 1: We begin with the latter part.

First we define some additional notation. For each N let:

$$Q = \frac{\hat{b}}{\hat{d} - g}\lambda = \frac{N\lambda s}{1 + (1 - 1/N)N\lambda s}$$

where $s = \hat{b}/\hat{d}$. Using this definition we can rewrite (5) as:

$$\beta\lambda^2 - 2Q + 1 = 0$$

Solving this for λ then yields:

$$\lambda = \sqrt{\frac{2Q - 1}{\beta}}.$$

Note that, if we find a $\lambda \in (0, 1)$ that satisfies (5) the above implies that $1/2 < Q < (1 + \beta)/2$. Now, define the following functions over $1/2 \leq q \leq$

$(1+\beta)/2$:

$$\Lambda(q) = \sqrt{\frac{2q-1}{\beta}} \in [0,1]$$

$$s(q) = \left.\frac{\hat{b}}{\hat{d}}\right|_{\lambda=\Lambda(q)}$$

$$\eta(q) = \frac{Ns(q)}{\sqrt{\beta}} > N/\sqrt{\beta} > N \tag{A1}$$

$$y(q) = \eta(q)\sqrt{2q-1} = N\Lambda(q)s(q)$$

$$\gamma(q) = \frac{y(q)}{1+(1-1/N)y(q)} = \frac{N\Lambda(q)s(q)}{1+(1-1/N)N\Lambda(q)s(q)}$$

$$k(q) = q - \gamma(q).$$

Suppose we knew an equilibrium value of λ. This would imply a value for Q so that $\lambda = \Lambda(Q)$. Then, by construction, $\hat{b}/\hat{d} = s(Q)$, $N\lambda\hat{b}/\hat{d} = y(Q)$ and $Q = \gamma(Q)$. That is, the Q corresponding to an equilibrium will be a fixed point of $\gamma(\,\cdot\,)$. Also, if q^* is a fixed point of $\gamma(\,\cdot\,)$ it is easily verified that $\Lambda(q^*)$ satisfies the equilibrium conditions. So equilibria will correspond one-to-one with fixed points of $\gamma(\,\cdot\,)$ or, equivalently, to points at which $k(q^*) = 0$. Note that all the above functions are continuous over the range of q and all are nondecreasing in q except $k(\,\cdot\,)$. With this notation, we can now turn to the proof.

The strategy of the proof is to show that there is a unique $q^* \in [1/2, (1+\beta)/2]$ such that $k(q^*) = 0$ and $y(q^*) < 1$. Now, we can write:

$$k(q) = \frac{q - \eta(q)\sqrt{2q-1}[1-(1-1/N)q]}{1+(1-1/N)\eta(q)\sqrt{2q-1}}$$

and this will be zero if and only if the numerator is zero. Define, for $1/2 \leq q \leq (1+\beta)/2$ and $\eta > N/\sqrt{\beta}$ (since $\eta(q) > N/\sqrt{\beta}$):

$$\mu(q,\eta) = q - \eta\sqrt{2q-1}[1-(1-1/N)q].$$

The following properties of μ are important: μ is continuous, strictly convex in q, and strictly decreasing in η. By construction, an equilibrium corresponds to a point q^* such that $\mu(q^*, \eta(q^*)) = 0$. Note also that the composite function $\mu(q, \eta(q))$ is continuous.

We next show an equilibrium exists. First:

$$\mu(1/2, \eta(1/2)) = 1/2 > 0.$$

Also:

$$\begin{aligned}\mu\left(\frac{1+\beta}{2}, \eta\right) &= \frac{1+\beta}{2} - \eta\sqrt{\beta}\left[1 - \left(1 - \frac{1}{N}\right)\frac{1+\beta}{2}\right] \\ &< \frac{1+\beta}{2} - N\left[1 - \left(1 - \frac{1}{N}\right)\frac{1+\beta}{2}\right], \ \forall \eta > \frac{N}{\sqrt{\beta}}\end{aligned}$$

so

$$\mu\left(\frac{1+\beta}{2}, \eta\left(\frac{1+\beta}{2}\right)\right) < \frac{1+\beta}{2}(1 + N - 1) - N = N\left(\frac{1+\beta}{2} - 1\right) < 0$$

since $\beta < 1$. By continuity, there exists $q^* \in (1/2, (1+\beta)/2)$ such that $\mu(q^*, \eta(q^*)) = 0$.

We now show q^* is unique. Take arbitrary $q \in (1/2, q^*)$. Since $q^* < (1+\beta)/2$, take $\alpha \in (0,1)$ such that $q^* = \alpha q + (1-\alpha)(1+\beta)/2$. By convexity:

$$\mu(q^*, \eta(q^*)) = 0 < \alpha\mu(q, \eta(q^*)) + (1-\alpha)\mu\left(\frac{1+\beta}{2}, \eta(q^*)\right).$$

The second term on the right-hand side is negative (since the above argument showed this was true for arbitrary η), so the first must be positive. Therefore $\mu(q, \eta(q^*)) > 0$. But $\eta(q)$ is increasing in q so $\eta(q^*) > \eta(q)$. Since μ is decreasing in η, this implies $\mu(q, \eta(q)) > \mu(q, \eta(q^*)) > 0$ so $\mu(q, \eta(q)) > 0$ for all $q \in [1/2, q^*)$. That $\mu(q, \eta(q))$ is not zero for $q \in (q^*, (1+\beta)/2]$ is now obvious since otherwise the same argument would imply $\mu(q^*, \eta(q^*)) > 0$.

Turning to the former part, we first show that $N\lambda\hat{b}/\hat{d} < 1$. Note that $N\lambda\hat{b}/\hat{d} = y(q^*)$ and if $q^* = \gamma(q^*) < N/(2N-1)$ it must be that $y(q^*) < 1$. So it suffices to show $q^* < N/(2N-1)$. If $N/(2N-1) > (1+\beta)/2$ we are done. Otherwise:

$$\begin{aligned}\mu\left(\frac{N}{2N-1}, \eta\left(\frac{N}{2N-1}\right)\right) &= \frac{N}{2N-1} - \eta\left(\frac{N}{2N-1}\right)\frac{1}{\sqrt{N-1}}\left[1 - \frac{N-1}{N}\cdot\frac{N}{2N-1}\right] \\ &= \frac{N}{2N-1}\frac{\sqrt{2N-1} - \eta\left(\dfrac{N}{2N-1}\right)}{\sqrt{2N-1}} \\ &< \frac{N}{2N-1}\frac{\sqrt{2N-1} - N}{\sqrt{2N-1}} \\ &\leq 0\end{aligned}$$

where the inequalities hold since $\eta > N$ and $N/(2N-1) \leq 1$, $N \geq \sqrt{2N-1}$.[13]

Finally, solving (8) for f and using (10) yields:

$$f = \frac{N-1}{N}\hat{a} - \frac{N-1}{N}\hat{b}(1-\lambda)\bar{p} + h\bar{c}$$

Substituting this, g and h into $\bar{p}$ and solving yields:

$$\bar{p} - \bar{c} = \frac{\hat{a} - (\hat{b} - \hat{d})\bar{c}}{\hat{b}(1 + N - (N-1)\beta\lambda) - \hat{d}(1+\beta)}. \tag{A2}$$

It is straightforward to see that as long as $\hat{a} > (\hat{b} - \bar{d})\bar{c}$ it must be the case that $\bar{p} > \bar{c}$.

Now we show that there exist $\varepsilon_L, \varepsilon_H$ where $p_t, X_t > 0$. Define

$$p_L = \bar{p} - N\frac{\lambda}{1-\lambda}\frac{\hat{b}}{\hat{d}}\varepsilon_L \tag{A3}$$

$$p_H = \bar{p} + N\frac{\lambda}{1-\lambda}\frac{\hat{b}}{\hat{d}}\varepsilon_H. \tag{A4}$$

It is easy to show that if $p_t \in [p_L, p_H]$ then $p_{t+1} \in (p_L, p_H)$. By induction, if $p_t \in [p_L, p_H]$ then $p_{t+k} \in (p_L, p_H)$ for any $k > 0$. Thus by bounding the errors we can ensure that prices set by the Euler equation will never be negative and will be lower than some price at which the households would always choose positive consumption in both periods. ■

Proof of Corollary 1: Since $1/2 < Q < N/(2N-1)$, Q must tend to $1/2$ as N tends to infinity. Since $\lambda = \Lambda(Q)$ and this function is continuous, λ tends to $\Lambda(1/2) = 0$.

Note Q is a one-to-one, continuous function of $y = N\lambda s$, so the fact that Q converges implies y converges to some limit point y'. But then it must be that $1/2 = y'/(1+y')$. Solving this yields $y' = 1$.

Rearranging (8) and substituting (13), we get

$$f = \frac{N-1}{N}\hat{a} - \frac{N-1}{N}\hat{b}(1-\lambda)\bar{p} + \frac{N-1}{N}\frac{N\lambda\hat{b}}{\hat{d}}\hat{b}\bar{c}.$$

Since we have already shown that $\lambda^* \to 0$ and $N\lambda^*\hat{b}/\hat{d} \to 1$, in the limit as $N \to \infty$, $f \to \hat{a} - \hat{b}(\bar{p} - \bar{c})$ or in other words, $\hat{a} - f \to \hat{b}(\bar{p} - \bar{c})$. It is straightforward to see that $h \to \hat{b}$ and $\hat{d} - g \to 0$ and therefore $\bar{p} \to \bar{c}$. ■

[13] This can easily be derived from the fact that $(N-1)^2 \geq 0$.

Proof of Proposition 1: Consider the case where $d > 0$.[14] The determinant of the matrix on the left hand side of (22) is $D = 1 - \theta_1(\theta_3\theta_6 + \theta_5)$. Now:

$$\theta_1 = \frac{m}{\sqrt{2q-1}} > \sqrt{2N-1}$$

since $m > 1$ and $q < N/(2N-1)$.

$$\theta_5 = \frac{Ns}{[1 + (1 - 1/N)N\lambda s]^2} > \frac{N}{[(2N-1)/N]^2} = \frac{N^3}{(2N-1)^2}$$

since $s > 1$ and $N\lambda s < 1$. Therefore $\theta_1\theta_5 > N^3/(2N-1)^{3/2}$. It is possible to show that the last expression equals one when $N = 1$ and is greater than one for $N > 1$. This guarantees that D is negative.

We can now sign the partial derivatives of λ using Cramer's rule:

$$\frac{d\lambda}{dN} = \frac{1}{D}\begin{vmatrix} 0 & 0 & -\theta_1 \\ 0 & 1 & 0 \\ \theta_7 & -\theta_6 & 1 \end{vmatrix} = \frac{\theta_1\theta_7}{D} < 0$$

$$\frac{d\lambda}{dm} = \frac{1}{D}\begin{vmatrix} \theta_2 & 0 & -\theta_1 \\ 0 & 1 & 0 \\ 0 & -\theta_6 & 1 \end{vmatrix} = \frac{\theta_2}{D} < 0$$

$$\frac{d\lambda}{dr} = \frac{1}{D}\begin{vmatrix} 0 & 0 & -\theta_1 \\ \theta_4 & 1 & 0 \\ 0 & -\theta_6 & 1 \end{vmatrix} = \frac{\theta_1\theta_4\theta_6}{D} < 0$$

In order to get some idea as to the behavior of the term $N\lambda\hat{b}/\hat{d}$, we will need to get similar comparative static results on s. These are:

$$\frac{ds}{dm} = \frac{1}{D}\begin{vmatrix} 1 & \theta_2 & -\theta_1 \\ -\theta_3 & 0 & 0 \\ -\theta_5 & 0 & 1 \end{vmatrix} = \frac{\theta_2\theta_3}{D} < 0$$

$$\frac{ds}{dr} = \frac{1}{D}\begin{vmatrix} 1 & 0 & -\theta_1 \\ -\theta_3 & \theta_4 & 0 \\ -\theta_5 & 0 & 1 \end{vmatrix} = \frac{\theta_4(1 - \theta_1\theta_5)}{D} > 0$$

follows from the fact that $\theta_1\theta_5 > 1$.

$$\frac{ds}{dN} = \frac{1}{D}\begin{vmatrix} 1 & 0 & -\theta_1 \\ -\theta_3 & 0 & 0 \\ -\theta_5 & \theta_7 & 1 \end{vmatrix} = \frac{\theta_1\theta_3\theta_7}{D} < 0$$

[14] The case where $d < 0$ is analogous but requires $\lambda = -\sqrt{(2q-1)/\beta}$.

Now, differentiating $y = N\lambda s$ with respect to m, e and N yields:

$$\frac{dy}{dm} = Ns\frac{d\lambda}{dm} + N\lambda\frac{ds}{dm} < 0$$
$$\frac{dy}{dr} = Ns\frac{d\lambda}{dr} + N\lambda\frac{ds}{dr} < 0$$

Finally, to sign dy/dN, first note that:

$$\begin{aligned}\frac{dy}{dN} &= \lambda s + N\lambda\frac{ds}{dN} + Ns\frac{d\lambda}{dN} \\ &= \lambda s - N\lambda\frac{\theta_1\theta_3\theta_7}{|D|} - Ns\frac{\theta_1\theta_7}{|D|}\end{aligned}$$

where $|D| = \theta_1(\theta_3\theta_6 + \theta_5) - 1 > 0$. Factoring out $|D|$, we get:

$$\frac{dy}{dN} = \frac{1}{|D|}(\theta_1[\theta_3(\theta_6\lambda s - \theta_7 N\lambda) + \theta_5\lambda s - \theta_7 Ns] - \lambda s)$$

Using the definition of the θs:

$$\frac{dy}{dN} = \frac{1}{|D|}\left(\theta_1\left[\theta_3\left(\frac{N\lambda^2 s}{H^2} - \frac{N\lambda^2 s(1-\lambda s)}{H^2}\right) + \frac{N\lambda s^2}{H^2} - \frac{N\lambda s^2(1-\lambda s)}{H^2}\right] - \lambda s\right)$$

where $H = 1 + (N-1)\lambda s$. So,

$$\begin{aligned}\frac{dy}{dN} &= \frac{1}{|D|}\left(\theta_1\left[\theta_3\frac{N\lambda^3 s^2}{H^2} + \frac{N\lambda^2 s^3}{H^2}\right] - \lambda s\right) \\ &= \frac{1}{|D|}\left(\theta_3\frac{N\lambda^2 s^2}{\beta H^2} + \frac{N\lambda s^3}{\beta H^2} - \lambda s\right)\end{aligned}$$

since $\theta_1 = 1/\beta\lambda$. Factoring out $\lambda s/H^2$ yields:

$$\frac{dy}{dN} = \frac{\lambda s}{|D|H^2}\left(\frac{\theta_3 N\lambda s}{\beta} + \frac{Ns^2}{\beta} - H^2\right)$$

This is positive whenever:

$$\frac{Ns^2}{\beta} > H^2$$

or

$$\frac{s}{\sqrt{\beta}}\sqrt{N} > H.$$

Now:

$$
\begin{aligned}
\frac{s}{\sqrt{\beta}}\sqrt{N} - H &> \sqrt{N} - H \text{ (since } \frac{s}{\sqrt{\beta}} > 1) \\
&= \sqrt{N} - 1 - \frac{N-1}{N} N\lambda s \\
&= \sqrt{N} - 1 - \frac{N-1}{N} y \\
&> \sqrt{N} - 1 - \frac{N-1}{N} \text{ (since } y < 1) \\
&= \frac{N\sqrt{N} - 2N + 1}{N}
\end{aligned}
$$

The term in the numerator is negative for N between one and two. However, it is positive for $N = 3$ and is strictly increasing in N for $N \geq 3$, so dy/dN is positive for $N \geq 3$. ■

Proof of Proposition 2: Let $s = \hat{b}/\hat{d}$. Define:

$$
\lambda(q) = \sqrt{\frac{2q-1}{\beta}}
$$

$$
f(q) = 1 - s\lambda(q)
$$

$$
g(q) = s[(2 + \beta''\lambda(q)^2)s\lambda(q) - (1 + \beta''\lambda(q)^2)]
$$

$$
r(q) = \frac{f(q)}{g(q)}
$$

and

$$
\gamma(q) = \frac{s + \beta r(q)}{sr(q) + 1}\lambda(q) = \frac{sg(q) + \beta f(q)}{sf(q) + g(q)}\lambda(q)
$$

By construction, a fixed point of $\gamma(q)$ for $q \in [1/2, (1+\beta)/2]$ gives us an equilibrium of the optimal tax problem. However, γ has a discontinuity at $q = 1/2$. To fix this, note that:

$$
sf(q) + g(q) = \lambda(q)[\beta'' s^2 \lambda(q)^2 - s\beta''\lambda(q) + s^2]
$$

Using this:

$$
\gamma(q) = \frac{sg(q) + \beta f(q)}{\beta'' s^2 \lambda(q)^2 2 - s\beta''\lambda(q) + s^2}
$$

This removes the discontinuity. Moreover, the denominator is strictly positive. It is quadratic in λ and positive at $\lambda = 0$. Since its roots are complex, it must always be positive for all λ.

Let

$$k(q) = q - \gamma(q) = \frac{q[\beta'' s^2 \lambda(q)^2 2 - s\beta''\lambda(q) + s^2] - sg(q) - \beta f(q)}{\beta'' s^2 \lambda(q)^2 2 - s\beta''\lambda(q) + s^2}.$$

Since the denominator is positive, the condition for equilibrium can be written as $\mu(q^*) = 0$ where:

$$\mu(q) = q[\beta'' s^2\lambda(q)^2 - s\beta''\lambda(q) + s^2] - s^2[(2+\beta''\lambda(q)^2)s\lambda(q) - 1 - \beta''\lambda(q)^2] - \beta[1 - s\lambda(q)].$$

Now,

$$\mu\left(\frac{1}{2}\right) = \frac{3}{2}s^2 - \beta > 0.$$

Also,

$$\mu\left(\frac{1+\beta}{2}\right) = -(2+\beta'')s^3 + \frac{3+\beta}{2}(\beta''+1)s^2 - \left(\frac{1+\beta}{2}\beta'' - \beta\right)s - \beta.$$

This is $-(1-\beta)/2 < 0$ at $s = 1$. Its derivative with respect to s is:

$$\frac{\partial\mu((1+\beta)/2)}{\partial s} = -3(2+\beta'')s^2 + (3+\beta)(\beta''+1)s - \frac{1+\beta}{2}\beta'' + \beta.$$

It is straightforward to show this is also negative at $s = 1$. Finally,

$$\frac{\partial^2\mu((1+\beta)/2)}{\partial s^2} = -6(2+\beta'')s + (3+\beta)(\beta''+1).$$

This last equation is negative at $s = 1$ and decreasing in s, so both derivatives are negative for $s > 1$, which implies $\mu((1+\beta)/2) < 0$. By continuity, there exists $q^* \in (1/2, (1+\beta)/2)$ such that $\mu(q^*) = 0$. ■

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